VARIETIES OF *A*-LOOP WITH FINITENESS PROPERTIES

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Abstract: Some varieties of *A*-loops, such that all their loops satisfy the property to be residual finite (nilpotent, or solvable) are describes.

By *loop* we mean an algebra *L* with an operation of multiplication $\cdot$ and two operations of division /, \, where there is an element $e \in L$, that for any elements $x, y \in L$ the following equalities hold true

$$x(x \setminus y) = x \setminus (xy) = (y / x)x = (yx) / x = y,$$

$$ex = xe = x.$$

Let *a* be a certain element of the loop *L*. The *left* and *right translations* L_a and R_a are defined by the equalities

$$xL_a = ax,\ xR_a = xa;\ x \in L.$$

The group $\mathrm{M}(L)$, generated by all translations of the form L_a and R_a, $a \in L$, is called *multiplicative group* of the loop *L*. The element $\alpha \in \mathrm{M}(L)$ is called an *internal substitution*, if $e\alpha = e$, i.e. α applies the unit *e* of the loop *L* on itself. All internal substitutions of the loop *L* form a subgroup in $\mathrm{M}(L)$, called the *group of internal substitutions* of the loop *L*, which will be denoted by *J(L)*. It is known (see [7]) that the group *J(L)* is generated by all substitutions of the form

$$R_{x,y} = R_x R_y R_{xy}^{-1},\quad L_{x,y} = L_y L_x L_{xy}^{-1},\quad T_x = R_x L_x^{-1}$$

$$(x, y \in L).$$

The subloop *H* of the loop *L* is *normal* when and only when $H\alpha = H$ for any $\alpha \in J(L)$ (see [7]-[8]).

Let be L a loop and $x, y, z \in L$ any its elements.

The element

$$[x, y, z] = x \setminus ((xy \cdot z)/(yz))$$

is called the *right associator* of the elements $x, y, z \in L$.

The element

$$(x, y, z) = ((xy) \setminus (x \cdot yz)) / z$$

is called the *left associator* of the elements $x, y, z \in L$.

The element $[x, y] = x/(y/xy)$ is called the *commutator* of elements $x, y \in L$.

It is known that for any elements $x, y, z \in L$ the following equalities hold true

$$xR_{y,z} = x[x, y, z],\ zL_{y,x} = (x, y, z)z,$$

$$xT_y = x[x, y].$$

For any subset H of a loop L will note by $[H, L]$ the subloop L, generated by all its commutators, left and right associators of the form $[h, x, y], (x, y, h),\ (h, x)$, where $h \in H,\ x, y \in L$, and will called the *mutually associator-commutator* or the *mutually associator-commutator subloop* of *H* and *L*. In particular, $[L, L]$ is called the *associator-commutator*, or the *associator-commutator subloop*, or *derivate subloop* of *L* and it is denoted by L'.

Let be

$$L_0 \supseteq L_1 \supseteq \ldots \supseteq L_n \supseteq \ldots$$

the *lower central series* of the loop *L*, i.e. $L_0 = L$, and for any *n>0* L_n / L_{n+1} is the subloop from the centre of the factor loop L / L_{n+1}. The loop *L* is called *nilpotent* (or *centrally nilpotent)* if there is such a natural number *n* that $L_n = \{e\}$. The least natural number *n* for which $L_n = \{e\}$ is called the *nilpotence class* of *L*.

Let be *L* a loop and

$$L = L^{(0)} \supseteq L' \supseteq L'' \supseteq ... \supseteq L^{(n)} \supseteq ...$$

the *derived series* of *L*, that is

$$L = L^{(0)}, L^{(n+1)} = [L^{(n)}, L^{(n)}],\ n \geq 1.$$

The loop *L* is called *solvable* (or *central solvable*) if there is such a natural number *m* for which $L^{(m)} = \{e\}$. The least natural number *n* for which $L^{(n)} = \{e\}$ is called the *solvable class* of *L*.

We say that the loop *L* is *finite residual* (*nilpotent* or *solvable*), if for any non-unite element $x \in L$ there is an homomorfism φ_x from *L* on a finite loop (nilpotent or solvable) such that $x\varphi_x \neq e$. A loop *L* is called *locally nilpotent* if any of its generated by a finite number of elements subloop is nilpotent.

If any internal substitution of a loop is automorphism, then it is called *A*-loop. As example of *A*-loops are commutative Moufang loops, i. e. loops whose elements verify equalities

$$xy = yx, \quad xy \cdot zx = x(yz) \cdot x.$$

Other terms and notions related to loops can be found in [1].

An important result from the theory of universal algebras is the classical theorem of Birkhoff [2, p.362]: *a variety of algebras is generated by all its finite algebras if and only if its free loops are residual finite.*

In [3] are shown the following statements.

Theorem 1. *Let be K a [quasi]variety of commutative Moufang loops. The following conditions are equivalent:*

- *(i) the class K consists only of residual-solvable loops;*
- *(ii) each loop of K is different from its associator;*
- *(iii) for each loop* $L \in K$ *it takes place* $\bigcap_{i=1}^{\infty} L^{(i)} =<1>$, *where* L^{i} *is the upper central series term.*

Theorem 2. *Let be K a [quasi]variety of commutative Moufang loops. The following conditions are equivalent:*

- *(i) the class K consists only of residual-solvable loops;*
- *(ii) each loop of K is different from its associator;*
- *(iii) for each loop* $L \in K$ *it takes place* $\bigcap_{i=1}^{\infty} L_{(i)} =<1>$, *where* $L_{(i)}$ *is the lower central series term.*

Corollary 1. *There is 3-periodic commutative Moufang loops containing a subloop that coincide with its asociator in given loop.*

We prove that the following is true:

Theorem 3. *Let be K a variety of locale nilpotent A-loops. Then all A-loops of a variety K are residual finite if and only if K is generated by a finite abelian group.*

First of all we will prove the following statement:

Lemma 1. *Any variety of a local nilpotent A-loop is generated by its finite loops.*

Proof. Indeed, let be *K* a variety referred to in the statement of Lemma and let be $F_\omega = F_\omega(K)$ a numerable rank *K*-free loop. We denote with *N* the subvariety of *K* generated by all *K*-finite loops. Let be *H* an arbitrary subloop of the F_ω loop, generated by a finite number of elements. So *A*-loop *H* is nilpotent and finitely generated. Then, by Theorem 1 [4], *H* is finite residual.

From here, according to the theorem 6 of [2], *H* is isomorphic included in a cartesian product $\prod_{i \in I} H_i$ of homomorfe images of finite loops H_i of the *H*.

As the loops $H_i, i \in I$ belong to the variety *N*, then *H* also belongs to *N*.

So we can conclude that every finitely generated subloop of *K*-free loop F belongs to the variety N, i. e. in F is true all the identities that define N and so $F \in N$. Therefore we obtain $K = V(F) \subseteq N$, so $K = N$.

From Lemma 1 and cited above Birkhoff's theorem immediately follows

Corollary 2. *A free loop of any variety of local nilpotent A-loop is residual finite.*

Proof of Theorem 3. *Sufficiency.* Indeed, let be $K = V(L)$, where *L* is the finite abelian group and let be *H* a certain group of *K*. According to [5], the variety *K* coincides with quasivariety *Q(L)* generated by abelian group *L*.

Then, by Theorem 8 [2, p 294], the group *H* is a subgroup of the cartesian product of groups isomorphic to *L*. But this also means that *H* is finite approximate.

Necessity. At the beginning we show that the variety *K* consists only of approximate finite *A*-loops, then each *A*-loop from *K* is abelian group. Suppose that some loops in *K* are nonassociative or noncommutative. Then, based on Lemma 1, a finite nonassociative, or noncommutative and nilpotent loop there exists in *K*. This means that variety *K* contains nonassociative or noncommutative *A*-loops with nilpotent class 2.

Let be *F* a free nilpotent of class 2 *A*-loop with strictly numerical rank of the variety formed by all *K*-loops with nilpotent class ≤ 2 and $x_i, i \in N = \{1,2,...\}$ are its free generators. And let be *H* a subloop generated in *F* by the union of the sets

$$\{[x_1, x_2, x_3] \cdot [x_i, x_j, x_k]^{-1} | 1 \leq i < j < k,$$

$$(i, j, k) \in N^3\}$$

and

$$\{[x_1, x_2] \cdot [x_i, x_j]^{-1} \mid 1 \leq j < k, (i, j) \in N^2\}.$$

Then equality relations

$$[x_1 H, x_2 H, x_3 H] = [x_i H, x_j H, x_k H],$$

where $1 \leq i < j < k$ are true in the factor *A*-loop $L = F / H$, and $[x_1 H, x_2 H] = [x_i H, x_j H]$, where $1 \leq i < j$. From here and inequalities

$$[x_1 H, x_2 H, x_3 H] \cdot [x_i H, x_j H, x_k H]^{-1} \neq 1H,$$

$$[x_1 H, x_2 H] \cdot [x_i H, x_j H]^{-1} \neq 1H$$

results

$$[x_1 H, x_2 H, x_3 H] \neq 1H, \ [x_1 H, x_2 H] \neq 1H$$

and, so, factor *A*-loop *F/H* is nonassociative or noncommutative.

Now let be φ an homomorphism from *A*-loop *L* in some finite *A*-loop *H*. Then, there exist some $p < q < r$, such that the images of elements $x_p H$, $x_q H$ and $x_r H$ from L by homomorphism φ are the same. So we have

$$(x_p H)\varphi = (x_q H)\varphi = (x_r H)\varphi,$$

and obtain

$$([x_1 H, x_2 H, x_3 H])\varphi = ([x_p H, x_q H, x_r H])\varphi =$$

$$= [(x_p H)\varphi, (x_q H)\varphi, (x_r H)\varphi] = 1$$

and

$$([x_1 H, x_2 H])\varphi = ([x_p H, x_q H])\varphi = 1.$$

From this and the relations of equality, indicated in factor-loop F / H, it follows equalities

$$[(x_i H)\varphi, (x_j H)\varphi, (x_k H)\varphi] = 1$$

and

$$[(x_i H)\varphi, (x_j H)\varphi] = 1$$

for any $1 \leq i < j < k$.

It results that the image of the associator-commutator L' by homomorphism φ is equal to the unity loop. Therefore, we conclude that for any homomorphism φ from *A*-loop *L* on a finite *A*-loop, the images of the elements of L' are equal to unity. But this means that *A*-loop *L* is not residual finite. On the other hand, $L = F / H \in K$ and, by definition, *L* is residual finitely, contradiction. Thus, all loops of a variety K are associative and commutative, so they are abelian groups.

Now we show that exponent of a variety *K* is finite. Let suppose it does not. Then in *K* is contained the infinite cyclic group *Z*, so, for a prime number *p*, a variety *K* contains all cyclic *p*-groups Z_{p^n}, $n \in N$ as homomorphe images of the infinite cyclic group *Z*. From here, in view of G. Birkhoff's Theorem (on the structure of varieties), [6] it is easily to see that K contains quasiciclic group p^{∞}. But this is not true, because the quasiciclic group p^{∞} is not residual finite.

Therefore, the exponent of *K* variety, is a finite number *m*, that is, in each A-loop of *K* is true the identity $x^m = 1$. So, K contains a finite set of finite cyclic nonisomorphe groups, the product of which generates a variety *K*. The theorem is proved.

According to [5], a finitely generated nilpotent A-loop has a finite basis of identities if and only if it is a finite abelian group. Therefore, from Theorem 3 we obtain the following

Corollary 3. *All loops of a variety of local nilpotent A-loops are residual finite, if and only if its finite free loop of rank* $r \geq 3$ *has a finite basis of quasiidentities.*

References

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