A new refinement of Jensen's inequality

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ABSTRACT. Using the Popoviciu's inequality, in this paper we present a new refinement of Jensen's inequality (see [1] and [2]).

Theorem 1. (Popoviciu, T.). If $f : I \to R$ $(I \subseteq R)$ is a convex function, $x_i \in I$, $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $k \in \{2, 3, ..., n-1\}$, then

$$\sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k}) f\left(\frac{\lambda_{i_1} x_{i_1} + ... + \lambda_{i_k} x_{i_k}}{\lambda_{i_1} + ... + \lambda_{i_k}}\right) \le$$

$$\le \binom{n-2}{k-2}\left(\left(\frac{n-k}{k-1}\right)\sum_{i=1}^{n} \lambda_i f(x_i) + \left(\sum_{i=1}^{n} \lambda_i\right) f\left(\frac{\lambda_1 x_1 + ... + \lambda_n x_n}{\lambda_1 + ... + \lambda_n}\right)\right) \quad (1)$$

Main Results

Theorem 2. (A refinement of Jensen's inequality). If $f : I \to R$ $(I \subseteq R)$, is a convex function $x_i, \lambda_i > 0$ $(i = 1, 2, ..., n)$, $k \in \{1, 2, ..., n\}$, then

$$\frac{\sum_{i=1}^{n} \lambda_i f(x_i)}{\sum_{i=1}^{n} \lambda_i} \ge \frac{n-k}{n-1} \frac{\sum_{i=1}^{n} \lambda_i f(x_i)}{\sum_{i=1}^{n} \lambda_i} + \frac{k-1}{n-1} f\left(\frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right) \ge$$

$$\ge \frac{1}{\binom{n-1}{k-1}\left(\sum_{i=1}^{n} \lambda_i\right)} \sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k}) f\left(\frac{\lambda_{i_1} x_{i_1} + ... + \lambda_{i_k} x_{i_k}}{\lambda_{i_1} + ... + \lambda_{i_k}}\right) \ge f\left(\frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right)$$

If f is concave, then holds the reverse inequalities.

Proof. Using the Jensen's inequality we obtain the following majorization

$$\frac{n-k}{k-1}\sum_{i=1}^{n} \lambda_i f(x_i) + \left(\sum_{i=1}^{n} \lambda_i\right) f\left(\frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right) \le$$

$$\le \frac{n-k}{k-1}\sum_{i=1}^{n} \lambda_i f(x_i) + \sum_{i=1}^{n} \lambda_i f(x_i) = \frac{n-1}{k-1}\sum_{i=1}^{n} \lambda_i f(x_i) \quad (2)$$

Using again the Jensen's inequality we obtain the following minorization:

$$\frac{\sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k}) f\left(\frac{\lambda_{i_1} x_{i_1} + ... + \lambda_{i_k} x_{i_k}}{\lambda_{i_1} + ... + \lambda_{i_k}}\right)}{\sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k})} \ge$$

$$\ge f\left(\frac{\sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k}) \frac{\lambda_{i_1} x_{i_1} + ... + \lambda_{i_k} x_{i_k}}{\lambda_{i_1} + ... + \lambda_{i_k}}}{\sum_{1 \le i_1 < ... < i_k \le n} (\lambda_{i_1} + ... + \lambda_{i_k})}\right) = f\left(\frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right)$$

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because

$$\sum_{1\leq i_1<\ldots<i_k\leq n}(\lambda_{i_1}+\ldots+\lambda_{i_k})=\binom{n-1}{k-1}\left(\sum_{i=1}^{n}\lambda_i\right)$$

So we obtain

$$\sum_{1\leq i_1<\ldots<i_k\leq n}(\lambda_{i_1}+\ldots+\lambda_{i_k})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\ldots+\lambda_{i_k}x_{i_k}}{\lambda_{i_1}+\ldots+\lambda_{i_k}}\right)\geq$$

$$\geq\binom{n-1}{k-1}\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)\tag{3}$$

Using (1) and (2) we get:

$$\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}\lambda_i f(x_i)+\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)\right)\leq$$

$$\leq\binom{n-2}{k-2}\frac{n-1}{k-1}\sum_{i=1}^{n}\lambda_i f(x_i)=\binom{n-1}{k-1}\sum_{i=1}^{n}\lambda_i f(x_i)\tag{4}$$

Using (1) and (3) we get:

$$\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}\lambda_i f(x_i)+\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)\right)\geq$$

$$\geq\sum_{1\leq i_1<\ldots<i_k\leq n}(\lambda_{i_1}+\ldots+\lambda_{i_k})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\ldots+\lambda_{i_k}x_{i_k}}{\lambda_{i_1}+\ldots+\lambda_{i_k}}\right)\geq$$

$$\geq\binom{n-1}{k-1}\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)\tag{5}$$

Dividing by $\binom{n-1}{k-1}\left(\sum\limits_{i=1}^{n}\lambda_i\right)$ from (4) and (5) we obtain the desired inequalities.

Remark. *If we consider*

$$y_k=\frac{n-k}{n-1}\cdot\frac{\sum\limits_{i=1}^{n}\lambda_i f(x_i)}{\sum\limits_{i=1}^{n}\lambda_i}+\frac{k-1}{n-1}f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)$$

then from Jensen's inequality we obtain that $y_1\geq y_2\geq\ldots\geq y_n$ which offer a lot of refinements of Jensen's inequality.

If we consider

$$z_k=\frac{1}{\binom{n-1}{k-1}\left(\sum\limits_{i=1}^{n}\lambda_i\right)}\sum_{1\leq i_1<\ldots<i_k\leq n}(\lambda_{i_1}+\ldots+\lambda_{i_k})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\ldots+\lambda_{i_k}x_{i_k}}{\lambda_{i_1}+\ldots+\lambda_{i_k}}\right)$$

then $z_1 \geq z_2 \geq ... \geq z_n$ which offer again a lot of refinements of Jensen's inequality.
Corollary 2.1. (A refinement of weighted arithmetical - geometrical means inequality).If $x_i, \lambda_i > 0\,(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i} \geq \prod_{1 \leq i_1 < ... < i_k \leq n}\left(\frac{\lambda_{i_1}x_{i_1} + ... + \lambda_{i_k}x_{i_k}}{\lambda_{i_1} + ... + \lambda_{i_k}}\right)^{\frac{\lambda_{i_1}+...+\lambda_{i_k}}{\binom{n-1}{k-1}\left(\sum\limits_{i=1}^{n}\lambda_i\right)}} \geq$$

$$\geq \prod_{i=1}^{n}\left(x_i^{\lambda_i}\right)^{\frac{n-k}{(n-1)\left(\sum\limits_{i=1}^{n}\lambda_i\right)}}\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)^{\frac{k-1}{n-1}} \geq \left(\prod_{i=1}^{n}x_i^{\lambda_i}\right)^{\frac{1}{\sum\limits_{i=1}^{n}\lambda_i}}$$

Proof. In Theorem 2 we consider $f(x) = \ln x$.
Corollary 2.2. (A refinement of power mean inequality.) If $x_i, \lambda_i > 0\ (i = 1, 2, ..., n)\,;\ s \geq p > 0$ and $k \in \{1, 2, ..., n\}$, then:

$$\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i^s}{\sum\limits_{i=1}^{n}\lambda_i}\right)^{\frac{1}{s}} \geq \left(\frac{n-k}{n-1}\cdot\frac{\sum\limits_{i=1}^{n}\lambda_i x_i^s}{\sum\limits_{i=1}^{n}\lambda_i} + \frac{k-1}{n-1}\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i^p}{\sum\limits_{i=1}^{n}\lambda_i}\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} \geq$$

$$\geq \left(\frac{1}{\binom{n-1}{k-1}\left(\sum\limits_{i=1}^{n}\lambda_i\right)}\sum_{1 \leq i_1 < ... < i_k \leq n}(\lambda_{i_1} + ... + \lambda_{i_k})\left(\frac{\lambda_{i_1}x_{i_1}^p + ... + \lambda_{i_k}x_{i_k}^p}{\lambda_{i_1} + ... + \lambda_{i_k}}\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} \geq$$

$$\geq \left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i^p}{\sum\limits_{i=1}^{n}\lambda_i}\right)^{\frac{1}{p}}$$

Proof. In Theorem 2 first we take $f(y) = y^t$ which is convex for $t \geq 1$, $\lambda_i, y_i > 0\ (i = 1, 2, ..., n)$ after then we consider $t = \frac{s}{p}$, and $y_i = x_i^p\ (i = 1, 2, ..., n)$.
Corollary 2.3. (A refinement of Minkowski's inequality). If $a_i, b_i > 0\ (i = 1, 2, ..., n)\,;\ p \geq 1$ and $k \in \{1, 2, ..., n\}$, then

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\frac{n-k}{k-1}\sum_{i=1}^{n}(a_i + b_i)^p + \frac{k-1}{n-1}\left(\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\frac{1}{\binom{n-1}{k-1}}\sum_{1 \leq i_1 < ... < i_k \leq n}\left(\left(a_{i_1}^p + ... + a_{i_k}^p\right)^{\frac{1}{p}} + \left(b_{i_1}^p + ... + b_{i_k}^p\right)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}a_1^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}b_1^p\right)^{\frac{1}{p}}$$

Proof. The function $f(y) = \left(1 + y^{\frac{1}{p}}\right)^p$ is concave, $y_i, \lambda_i > 0\ (i = 1, 2, ..., n)$. We take $y_i = x_i^p$ $(i = 1, 2, ..., n)$ and after then $\lambda_i = a_i^p$, $x_i = \frac{b_i}{a_i}\ (i = 1, 2, ..., n)$.
Corollary 2.4. (A refinement of Huygen's inequality). If $a_i, b_i, \lambda_i > 0\ (i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\left(\prod_{i=1}^{n}(a_i+b_i)^{\lambda_i}\right)^{\frac{1}{\sum_{i=1}^{n}\lambda_i}} \geq \left(\prod_{i=1}^{n}(a_i+b_i)^{\lambda_i}\right)^{\frac{n-k}{(n-1)\sum_{i=1}^{n}\lambda_i}} \left(\prod_{i=1}^{n}a_i^{\lambda_i}+\prod_{i=1}^{n}b_i^{\lambda_i}\right)^{\frac{k-1}{(n-1)\sum_{i=1}^{n}\lambda_i}} \geq$$

$$\geq \prod_{1\leq i_1<...<i_k\leq n}\left(\left(a_{i_1}^{\lambda_{i_1}}...a_{i_k}^{\lambda_{i_k}}\right)^{\frac{1}{\lambda_{i_1}+...+\lambda_{i_k}}}+\left(b_{i_1}^{\lambda_{i_1}}...b_{i_k}^{\lambda_{i_k}}\right)^{\frac{1}{\lambda_{i_1}+...+\lambda_{i_k}}}\right)^{\frac{\lambda_{i_1}+...+\lambda_{i_k}}{\binom{n-1}{k-1}\sum_{i=1}^{n}\lambda_i}} \geq$$

$$\geq \left(\prod_{i=1}^{n}a_i^{\lambda_i}+\prod_{i=1}^{n}b_i^{\lambda_i}\right)^{\frac{1}{\sum_{i=1}^{n}\lambda_i}}$$

Proof. The function $f(y)=\ln(1+e^y)$ is convex, $y_i,\lambda_i>0$ $(i=1,2,...,n)$. We consider $x_i=e^{y_i}$ $(i=1,2,...,n)$ and after then $x_i=\frac{b_i}{a_i}$ $(i=1,2,...,n)$.

Corollary 2.5. (A refinement of Hölder's inequality). If $a_i,b_i,\lambda_i>1$ $(i=1,2,...,n)$, $\alpha,\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, and $k\in\{1,2,...,n\}$, then

$$\sum_{i=1}^{n}\lambda_i a_i b_i \leq \left(\sum_{i=1}^{n}\lambda_i a_i^{\alpha}\right)\cdot$$

$$\cdot\left(\frac{1}{\binom{n-1}{k-1}\left(\sum_{i=1}^{n}\lambda_i a_i^{\alpha}\right)}\sum_{1\leq i_1<...<i_k\leq n}\left(\lambda_{i_1}a_{i_1}^{\alpha}+...+\lambda_{i_k}a_{i_k}^{\alpha}\right)\left(\frac{\lambda_{i_1}a_{i_1}b_{i_1}+...+\lambda_{i_k}a_{i_k}b_{i_k}}{\lambda_{i_1}a_{i_1}^{\alpha}+...+\lambda_{i_k}a_{i_k}^{\alpha}}\right)^{\beta}\right)^{\frac{1}{\beta}} \leq$$

$$\leq \left(\sum_{i=1}^{n}\lambda_i a_i^{\alpha}\right)\left(\frac{n-k}{n-1}\cdot\frac{\sum_{i=1}^{n}\lambda_i b_i^{\beta}}{\sum_{i=1}^{n}\lambda_i a_i^{\alpha}}+\frac{k-1}{n-1}\left(\frac{\sum_{i=1}^{n}\lambda_i a_i b_i}{\sum_{i=1}^{n}\lambda_i a_i^{\alpha}}\right)^{\beta}\right)^{\frac{1}{\beta}} \leq \left(\sum_{i=1}^{n}\lambda_i a_i^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\lambda_i b_i^{\beta}\right)^{\frac{1}{\beta}}$$

Proof. In Corollary 2.2 we take $s=1$, and $p\in(0,1)$ for $\mu_i\ y_i>0$ $(i=1,2,...,n)$. After then we consider $p+r=1$, $\mu_i=\lambda_i x_i$ $(i=1,2,...,n)$.
Again we substitute $y_i=\frac{z_i}{x_i}$ $(i=1,2,...,n)$ and finally $x_i=a_i^{\alpha}$, $z_i=b_i^{\beta}$, $r=\frac{1}{\alpha}$, $p=\frac{1}{\beta}$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Remark. *If in Corollary 2.5 we take* $\lambda_i=\frac{1}{n}$, $\alpha=\beta=2$ $(i=1,2,...,n)$ *then we obtain a refinement of Cauchy-Schwarz's inequality.*

Corollary 2.6. (A refinement of Young's inequality). If $a_i>0$, $\lambda_i>1$ $(i=1,2,...,n)$ such that $\sum_{i=1}^{n}\frac{1}{\lambda_i}=1$ and $k\in\{1,2,...,n\}$, then

$$\prod_{i=1}^{n}a_i \leq \frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<...<i_k\leq n}\left(\frac{1}{\lambda_{i_1}}+...+\frac{1}{\lambda_{i_k}}\right)\left(a_{i_1}...a_{i_k}\right)^{\frac{1}{\lambda_{i_1}}+...+\frac{1}{\lambda_{i_k}}} \leq$$

$$\leq \frac{n-k}{n-1}\sum_{i=1}^{n}\frac{a_i^{\lambda_i}}{\lambda_i}+\frac{k-1}{n-1}\prod_{i=1}^{n}a_i \leq \sum_{i=1}^{n}\frac{a_i^{\lambda_i}}{\lambda_i}$$

Proof. The function $f(y)=e^y$ is convex, therefore if $p_i>0$ $(i=1,2,...,n)$, $\sum_{i=1}^{n}p_i=1$, then from Theorem 2 we get:

$$e^{\sum_{i=1}^{n}p_i y_i} \leq \frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<...<i_k\leq n}\left(p_{i_1}+...+p_{i_k}\right)\cdot e^{\frac{p_{i_1}y_{i_1}+...+o_{i_k}y_{i_k}}{y_{i_1}+...+y_{i_k}}} \leq$$

$$\leq \frac{n-k}{n-1}\left(\sum_{i=1}^{n}p_i e^{y_i}\right)+\frac{k-1}{n-1}e^{\sum_{i=1}^{n}p_i y_i} \leq \sum_{i=1}^{n}p_i e^{y_i}$$

In this we take $p_i=\frac{1}{\lambda_i}$, $y=\ln a_i^{\lambda_i}$ $(i=1,2,...,n)$ and we obtain the desired result.

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