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A Note on a Class of Analytic Functions Defined by a Generalized Sălăgean Operator

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Abstract. The object of this paper is to obtain some inclusion relations regarding a new class $M^m(\lambda, \alpha, \eta)$ using the generalized Sălăgean operator.

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1 Introduction

We define the class of normalized analytic functions $\mathcal{A}_n$ as

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots\}, \tag{1.1}$$

$n \in \mathbb{N}^* = \{1, 2, \dots\}$, with $\mathcal{A}_1 = \mathcal{A}$.

F.M. Al-Oboudi in [1] defined, for a function in $\mathcal{A}_n$, the following differential operator:

$$D^0 f(z) = f(z) \tag{1.2}$$

$$D^1_\lambda f(z) = D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) \tag{1.3}$$

$$D^m_\lambda f(z) = D_\lambda (D^{m-1}_\lambda f(z)), \quad \lambda > 0. \tag{1.4}$$

When $\lambda = 1$, we get the Sălăgean operator [5].

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(u) \subset g(U)$.

To prove the main results we will need the following lemmas.

Lemma 1.1 (Hallenbeck and Ruschweyh [2]) *Let h be convex in U with $h(0) = a$, $\gamma \neq 0$ and* Re $\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma 1.2 (Miller and Mocanu [3]) *Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z)$$

where $\alpha > 0$ and n is a positive integer. If $p \in \mathcal{H}(U)$ with

$$p(z) = q(0) + p_n z^n + \dots$$

and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

2 Main results

Definition 2.1 Let $f \in \mathcal{A}$. We say that the function f is in the class $M^m(\lambda, \alpha, \eta)$, $\lambda > 0$, $\alpha \in \mathbb{R}$, $\eta \in [0, 1)$, $m \in \mathbb{N}$, if f satisfies the condition

$$\text{Re}\, [D_\lambda^m \widetilde{J}(\alpha, f; z)]' > \eta, \quad z \in U \tag{2.1}$$

where

$$\widetilde{J}(\alpha, f; z) = zJ(\alpha, f; z) \tag{2.2}$$

$$J(\alpha, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right). \tag{2.3}$$

Remark 2.1 We note that the class $M^m(1, \alpha, \eta)$ was studied in [4].

Theorem 2.1 *If $\alpha \in \mathbb{R}$, $\eta \in [0, 1)$ and $m \in \mathbb{N}$ then*

$$M^{m+1}(\lambda, \alpha, \eta) \subset M^m(\lambda, \alpha, \delta) \tag{2.4}$$

where

$$\delta = \delta(\lambda, \eta) = 2\eta - 1 + 2(1 - \eta)\frac{1}{\lambda}\beta\left(\frac{1}{\lambda}\right) \tag{2.5}$$

β being the Beta function

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt. \tag{2.6}$$

Proof. Let $f \in M^{m+1}(\lambda, \alpha, \eta)$. By using the properties of the operator D_λ^m, we have

$$D_\lambda^{m+1}\widetilde{J}(\alpha, f; z) = (1-\lambda)D_\lambda^m\widetilde{J}(\alpha, f; z) + \lambda z(D_\lambda^m\widetilde{J}(\alpha, f; z))' \tag{2.7}$$

If we denote by

$$p(z) = (D_\lambda^m\widetilde{J}(\alpha, f; z))' \tag{2.8}$$

where

$$p(z) = 1 + p_2z^2 + \dots, \quad p(z) \in \mathcal{H}[1, 2],$$

then after a short computation we get

$$(D_\lambda^{m+1}\widetilde{J}(\alpha, f; z))' = p(z) + \lambda zp'(z), \quad z \in U. \tag{2.9}$$

Since $f \in M^{m+1}(\lambda, \alpha, \eta)$, from Definition 2.1 we have

$$\text{Re}\,(D_\lambda^{m+1}\widetilde{J}(\alpha, f; z))' > \eta, \quad z \in U.$$

Using (2.9) we get

$$\text{Re}\,(p(z) + \lambda zp'(z)) > \eta$$

which is equivalent to

$$p(z) + \lambda zp'(z) \prec \frac{1 + (2\eta - 1)z}{1 + z} \equiv h(z). \tag{2.10}$$

From Lemma 1.1, with $\gamma = \dfrac{1}{\lambda}$, we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{1}{\lambda z^{1/\lambda}}\int_0^z \frac{1 + (2\eta - 1)t}{1 + t}t^{(1/\lambda)-1}dt.$$

The function q is convex and is the best $(1, 1)$-dominant.

Since

$$(D_\lambda^m\widetilde{J}(\alpha, f; z))' \prec 2\eta - 1 + \frac{2(1-\eta)}{\lambda} \cdot \frac{1}{z^{1/\lambda}}\int_0^z \frac{t^{(1/\lambda)-1}}{t+1}dt$$

it results that

$$\text{Re}\,(D_\lambda^m\widetilde{J}(\alpha, f; z))' > q(1) = \delta \tag{2.11}$$

where

$$\delta = \delta(\lambda, \eta) = 2\eta - 1 + 2(1-\eta)\frac{1}{\lambda}\beta\left(\frac{1}{\lambda}\right) \tag{2.12}$$

$$\beta\left(\frac{1}{\lambda}\right) = \int_0^1 \frac{t^{(1/\lambda)-1}}{t+1}dt. \tag{2.13}$$

From (2.11) we deduce that $f \in M^m(\lambda, \alpha, \delta)$ and the proof of the theorem is complete. □

Theorem 2.2 *Let $q(z)$ be a convex function, $q(0) = 1$, and let h be a function such that*

$$h(z) = q(z) + \lambda z q'(z), \quad \lambda > 0. \tag{2.14}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$(D_\lambda^{m+1} \widetilde{J}(\alpha, f; z))' \prec h(z) \tag{2.15}$$

then

$$(D_\lambda^m \widetilde{J}(\alpha, f; z))' \prec q(z) \tag{2.16}$$

and the result is sharp.

Proof. From (2.7), (2.8) and (2.15) we obtain

$$p(z) + \lambda z p'(z) \prec q(z) + \lambda z q'(z) \equiv h(z) \tag{2.17}$$

then, by using Lemma 1.2 we get

$$p(z) \prec q(z)$$

or

$$(D_\lambda^m \widetilde{J}(\alpha, f; z))' \prec q(z), \quad z \in U$$

and this result is sharp. □

Theorem 2.3 *Let q be a convex function with $q(0) = 1$ and let h be a function of the form*

$$h(z) = q(z) + z q'(z), \quad z \in U. \tag{2.18}$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$(D_\lambda^m \widetilde{J}(\alpha, f; z))' \prec h(z), \quad z \in U \tag{2.19}$$

then

$$\frac{D_\lambda^m \widetilde{J}(\alpha, f; z)}{z} \prec q(z) \tag{2.20}$$

and this result is sharp.

Proof. If we let

$$p(z) = \frac{D_\lambda^m \widetilde{J}(\alpha, f; z)}{z}, \quad z \in U$$

then we obtain

$$(D_\lambda^m \widetilde{J}(\alpha, f; z))' = p(z) + z p'(z), \quad z \in U.$$

The subordination (2.19) becomes

$$p(z) + z p'(z) \prec q(z) + z q'(z)$$

and from Lemma 1.2 we have (2.20). The result is sharp. □

References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., 27 (2004), 1429-1436.

[2] D.J. Hallenbeck and St. Ruschweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. 52(1975), 191-195.

[3] S.S. Miller, P.T. Mocanu, *On some classes of first order differential subordinations*, Michigan Math. J., 32(1985), 185-195.

[4] G.I. Oros, *Differential subordination defined by Sălăgean operator*, (to appear).

[5] G.S. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, 1013(1983), 262-372.