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A Note on a Second Order Functional Differential Equation

Mehran MAHDAVI

This note is a direct continuation of two preceding papers, jointly with C. Corduneanu [4], [5]. The topic to be discussed here is the existence of solutions for the functional differential equation

$$\frac{d}{dt}\left[\frac{dx(t)}{dt} - (Lx)(t)\right] = (Vx)(t)\ ,\ t \in R_+, \tag{1}$$

where L and V are causal operators acting on the function space $L^2_{loc}(R_+, R^n)$. It will be assumed that L is linear, while V is, in general, nonlinear. The case where V is also linear leads to an integral equation which can be treated by means of classical methods. The nonlinear case for V leads to a functional equation which has been investigated, under somewhat different conditions, in [4], [5].

1 A Cauchy Problem on R_+

We shall attach to the equation (1) the usual initial value data

$$x(0) = x^0 \in R^n\ ,\ \dot{x}(0) = v^0 \in R^n. \tag{2}$$

As shown in [4], [5], the Cauchy problem (1), (2) is, under assumptions to be formulated below, equivalent to the functional equation

$$x(t) = X(t,0)\,x^0 + \int_0^t X(t,s)\,v^0 ds + \int_0^t X(t,s)\int_0^s (Vx)(u)\,du\,ds, \tag{3}$$

on the half-axis R_+.

The term "equivalent" may have various meanings, according to the set of hypotheses accepted on the operators L and V. For instance, when we deal with the space $L^2_{loc}(R_+, R^n)$, one sees that $x(t)$ from (3) will be an absolutely continuous function (locally on R_+), while the intermediate equation

$$\dot{x}(t) - (Lx)(t) = v^0 + \int_0^t (Vx)(s)\,ds, \tag{4}$$

obtained by integrating both sides of (1) from 0 to $t > 0$, does not allow us to conclude anything about the second derivative $\ddot{x}(t)$. It has been assumed, in obtaining (4), that L has the fixed initial condition $(Lx)(0) = \theta \in R^n$. See, for instance, [1] for details.

There is an important special case of equation (4) when V is also linear. Then, using the representation

$$\int_0^t (Vx)(s)\,ds = \int_0^t k_0(t,s)\,x(s)\,ds\ ,\ t \in R_+, \tag{5}$$

one obtains the following (classical) integral equation:

$$x(t) = f(t) + \int_0^t k_1(t,s)\,x(s)\,ds\ ,\ t \in R_+, \tag{6}$$

where

$$f(t) = X(t,0)\,x^0 + \int_0^t X(t,s)\,v^0\,ds\ ,\ t \in R_+, \tag{7}$$

and

$$k_1(t,s) = \int_s^t X(t,u)\,k_0(u,s)\,du, \tag{8}$$

with

$$(t,s) \in \Delta = \{(t,s) : 0 \le s \le t < \infty\}. \tag{9}$$

Under continuity assumptions for the operators L and V, besides (5) we have

$$\int_0^t (Lx)(s)\,ds = \int_0^t k(t,s)\,x(s)\,ds\ ,\ t \in R_+, \tag{10}$$

with $k(t,s)$ belonging to the space $L^2_{loc}(\Delta,\, \mathcal{L}(R^n, R^n))$. Then $X(t,s)$ is given by

$$X(t,s) = I + \int_s^t \tilde{k}(t,u)\,du \tag{11}$$

in Δ, where $\tilde{k}(t,u)$ is the resolvent kernel corresponding to $k(t,u)$.

The validity of the representations (5) and (10) is discussed in the book [2] by C. Corduneanu, where more references are provided. Generally speaking, the properties of L and V imply properties for $k_0(t,s)$ and $k(t,s)$. Illustrations are also provided in [4], [5].

Sometimes, the relationships (5) and (10) are written in equivalent form

$$(Vx)(t) = \frac{d}{dt}\int_0^t k_0(t,s)\,x(s)\,ds,$$

and,

$$(Lx)(t) = \frac{d}{dt}\int_0^t k(t,s)\,x(s)\,ds,$$

the differentiation being meant *a.e.* on R_+.

To summarize the discussion above, we see that in case both L and V are continuous operators on $L^2_{loc}(R_+, R^n)$, the Cauchy problem (1), (2) is equivalent to the integral equation (6), with $f(t)$ and $k_1(t,s)$ given by (7) and (8).

In case L is linear and V is nonlinear, the Cauchy problem (1), (2) is equivalent to the functional equation (3).

2 The Linear Case

We shall consider the integral equation (6) in view of proving the existence of a unique solution on R_+. This will be achieved by showing that

$$k_1(t,s) \in L^2_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{12}$$

The property (12) implies the existence of the resolvent kernel associated to $k_1(t,s)$, denoted by $\tilde{k}_1(t,s)$ and satisfying the condition

$$\tilde{k}_1(t,s) \in L^2_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{13}$$

Then, the unique solution of (6) is represented by the resolvent formula

$$x(t) = f(t) + \int_0^t \tilde{k}_1(t,s)\, f(s)\, ds\ ,\ t \in R_+, \tag{14}$$

which is valid anytime $f(t) \in L^2_{loc}(R_+, R^n)$. This is a classical result which can be found in Tricomi's book [8].

Therefore, the Cauchy problem (1), (2) has a unique solution on R_+, this solution being as described in section 1 of this paper: it is *a.e.* differentiable on R_+, and such that $\dot{x}(t)-(Lx)(t)$ is also *a.e.* differentiable. Moreover, the solution admits the integral representation (14), with $f(t)$ an absolutely continuous map, on each finite interval of R_+.

The last statement about $f(t)$ follows from the fact that it is the solution of the Cauchy problem for the homogeneous equation $\dot{x}(t) = (Lx)(t)$, under initial data (2). Hence, it is locally absolutely continuous. See [4], [5]. In order to conclude the discussion on the problem (1), (2), in the linear case, we need to prove that the kernel $k_1(t,s)$ given by (8), is indeed satisfying (12). Using the representation (11) for $X(t,s)$, and (8) for $k_1(t,s)$, we obtain for $(t,s) \in \Delta$ the following equation:

$$k_1(t,s) = \int_s^t \left[I + \int_u^t \tilde{k}(t,v)\, dv \right] k_0(u,s)\, du. \tag{15}$$

Since both $\tilde{k}(t,s)$ and $k_0(u,s)$ belong to L^2_{loc}, on behalf of our assumptions, there remains to prove that, for each $T > 0$

$$\int_0^T \int_0^T |k_1(t,s)|^2\, dt\, ds < \infty. \tag{16}$$

Condition (16) will be secured if we prove

$$\int_s^t \int_u^t |\tilde{k}(t,v)|\, |k_0(u,s)|\, dv\, du \in L^2_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)), \tag{17}$$

because it can be easily seen that

$$\int_s^t k_0(u,s)\, du \in L^2_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)).$$

In order to assume the validity of (17), we shall rely on the elementary integral inequality,

$$\left(\int_s^t \int_u^t |\tilde{k}(t,v)|\,|k_0(u,s)|\,dv\,du\right)^2 \le \left(\int_s^t \int_u^t |\tilde{k}(t,v)|^2\,dv\,du\right)\left(\int_s^t \int_u^t |k_0(u,s)|^2\,dv\,du\right),$$

and since,

$$\left(\int_s^t \int_u^t |\tilde{k}(t,v)|^2\,dv\,du\right)\left(\int_s^t \int_u^t |k_0(u,s)|^2\,dv\,du\right) \le$$
$$\le \left(\int_0^T \int_0^T |\tilde{k}(t,v)|^2\,dv\,du\right)\left(\int_0^T \int_0^T |k_0(u,s)|^2\,dv\,du\right).$$

Hence,

$$\left(\int_s^t \int_u^t |\tilde{k}(t,v)|\,|k_0(u,s)|\,dv\,du\right)^2 \le \left(\int_0^T \int_0^T |\tilde{k}(t,v)|^2\,dv\,du\right)\left(\int_0^T \int_0^T |k_0(u,s)|^2\,dv\,du\right). \tag{18}$$

It is obvious that the last term in (18) is dominated by

$$T^2\left(\int_0^T |\tilde{k}(t,v)|^2\,dv\right)\left(\int_0^T |k_0(u,s)|^2\,du\right). \tag{19}$$

When we integrate on Δ the function (of t and s) appearing in (19), we certainly obtain a finite number, because both $\tilde{k}$ and k_0 are locally square integrable on Δ.

Therefore, the discussion carried above leads to the conclusion that the L^2-theory can be applied to equation (6) in Δ, due to the fact that $T > 0$ is arbitrary in our above considerations.

Summarizing the above discussion about equation (6), equivalent to the Cauchy problem (1), (2), we can state the following result.

Theorem 1. *Consider the Cauchy problem (1), (2) on the positive half-axis R_+. Assume L and V are linear, causal and continuous on the space $L^2_{loc}(R_+, R^n)$ with $(Lx)(0) = \theta \in R^n$, $x \in L^2_{loc}(R_+, R^n)$. Then the problem has a unique solution in the space $L^2_{loc}(R_+, R^n)$, also satisfying the integral equation (6). The solution to (1), (2) is locally absolutely continuous on R_+, and such that $\dot{x}(t) - (Lx)(t)$ is a.e. differentiable there.*

3 The Nonlinear Case; equation (3)

Taking into account notation (7), we can rewrite equation (3) in the form

$$x(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx)(u)\,du\,ds, \tag{20}$$

in which the (possible) nonlinearity is under the integration sign. Equation (20) has been investigated in the joint paper [4], under the basic assumption that the operator V satisfies a type of generalized Lipschitz condition. It turns out that something similar to [4] is working under the new hypotheses we shall enunciate below.

Namely, the Lipschitz type condition we will impose on the operator$V$ is the one naturally suggested by the L^2-norm, which means

$$\int_0^t |(Vx)(s) - (Vy)(s)|^2\, ds \le \lambda(t) \int_0^t |x(s) - y(s)|^2\, ds, \tag{21}$$

with $\lambda(t)$ a non-negative nondecreasing function on R_+.

Condition (21) is obviously implying the causality of the operator V on $L^2_{loc}(R_+, R^n)$.

Also, it is implied by the usual Lipschitz condition

$$|(Vx)(t) - (Vy)(t)| \le \lambda(t)\, |x(t) - y(t)|\ ,\ t \in R_+, \tag{22}$$

even in case (22) is satisfied only *a.e.*.

Besides (21), one more assumption will be made, thus time related to the operator L, which determines the Cauchy function $X(t,s)$ on Δ. Namley, following [4], we shall assume that $X(t,s)$ satisfies the so-called stability condition

$$\int_0^t |X(t,s)|\, ds \le M < \infty,\ t \in R_+. \tag{23}$$

This property is closely related to the boundedness of solutions of the equation $\dot{x}(t) - (Lx)(t) = f(t)$, under initial condition $x(0) = x^0 \in R^n$. The solution is given by the formula

$$x(t) = X(t,0)\, x^0 + \int_0^t X(t,s)\, f(s)\, ds\ ,\ t \in R_+, \tag{24}$$

which shows that all solutions of the linear equation above are bounded on R_+, iff $X(t,0)$ is bounded there, (23) takes place, while $f(t)$ is bounded on R_+.

We can now proceed with equation (20) by the iteration method, constructing the sequence of successive approximations by means of the formula

$$x_{m+1}(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx_m)(u)\, du\, ds, \tag{25}$$

starting with an arbitrary $x_0(t) \in L^2_{loc}(R_+, R^n)$.

One obtains from (25) the following recurrent relationship:

$$x_{m+1}(t) - x_m(t) = \int_0^t X(t,s) \int_0^s [(Vx_m)(u) - (Vx_{m-1})(u)]\, du\, ds, \tag{26}$$

which is reducing, on behalf of (23), to the inequality ($m \ge 1$)

$$|x_{m+1}(t) - x_m(t)| \le M \int_0^t |(Vx_m)(u) - (Vx_{m-1})(u)|\, du, \tag{27}$$

if one takes into account that for $s \in [0,t]$,

$$\sup \int_0^s |(Vx_m)(u) - (Vx_{m-1})(u)|\, du = \int_0^t |(Vx_m)(u) - (Vx_{m-1})(u)|\, du.$$

In order to use condition (21), we will process inequality (27) in the following manner:

$$|x_{m+1}(t) - x_m(t)|^2 \leq M^2 \left(\int_0^t du \right) \left(\int_0^t |(Vx_m)(u) - (Vx_{m-1})(u)|^2\, du \right),$$

and if we choose $T > 0$ arbitrary, then for $t \in [0,T]$ we can write

$$|x_{m+1}(t) - x_m(t)|^2 \leq M^2 T \int_0^t |(Vx_m)(u) - (Vx_{m-1})(u)|^2\, du. \tag{28}$$

Based on condition (21), there results from (28)

$$|x_{m+1}(t) - x_m(t)|^2 \leq M^2 T \lambda(T) \int_0^t |x_m(s) - x_{m-1}(s)|^2\, ds, \tag{29}$$

and denoting $M^2 T \lambda(T) = K(T) = K > 0$, we can rewrite (29) in the form

$$|x_{m+1}(t) - x_m(t)|^2 \leq K \int_0^t |x_m(s) - x_{m-1}(s)|^2\, ds, \tag{30}$$

valid for $t \in [0,T]$.

The recurrent inequality (30) can be processed in the usual way, and leads to the estimate

$$|x_{m+1}(t) - x_m(t)|^2 \leq A \frac{(Kt)^m}{m!},\ m \geq 1, \tag{31}$$

where $A = \sup\{|x_1(t) - x_0(t)| : t \in [0,T]\}$.

We notice the fact that all terms $x_m(t)$, $m \geq 1$, are absolutely continuous functions, locally on R_+. Hence, (31) shows that on each finite interval of R_+, the sequence of approximations $\{x_m(t) : m \geq 1\}$ is uniformly convergent. This implies that the limit

$$\lim x_m(t) = x(t)\ ,\ t \in R_+, \tag{32}$$

is a continuous function on R_+. Moreover, it is a solution of the equation (20), which implies its absolute continuity on each finite interval $[0,T] \subseteq R_+$. The uniqueness proof follows the usual pattern.

The discussion above justifies the validity of the following result.

Theorem 2. *Consider the Cauchy problem (1), (2), under the following assumptions on the operators L and V:*

(1) *L is a linear, causal and continuous operator on the space $L^2_{loc}(R_+, R^n)$.*

(2) *L satisfies the initial value property $(Lx)(0) = \theta \in R^n$, for any $x \in L^2_{loc}(R_+, R^n)$.*

(3) *The Cauchy matrix $X(t,s)$, associated with the operator L, satisfies condition (23).*

(4) *The operator V is acting on the space $L^2_{loc}(R_+, R^n)$, and satisfies the condition (21).*

Then, there exists a unique solution $x(t)$ of the problem (1), (2), defined on R_+. This solution also satisfies the integral equation (3). It is locally absolutely continuous on R_+, and satisfies equation (1) *a.e.* on R_+.

The proof of Theorem 2 has been provided above, using the successive approximations. Fixed points method can be also helpful.

Remark. The condition (21) for V implies both causality and continuity in (L^2_{loc}) of this operator.

4 Some Final Remarks

In a joint paper with C. Corduneanu [3], we have investigated a second order neutral functional equation of the form

$$\frac{d}{dt}\left[\dot{x}(t) + (L\dot{x})(t)\right] = (Vx)(t), \tag{33}$$

under conditions similar to those used above, but in spaces of continuous functions.

If we denote

$$\dot{x}(t) = y(t) \Longrightarrow x(t) = x^0 + \int_0^t y(s)\,ds\ ,\ t \in R_+, \tag{34}$$

then (33) can be rewritten as a first order neutral equation, namely

$$\frac{d}{dt}\left[y(t) + (Ly)(t)\right] = (V_1 y)(t)\ ,\ t \in R_+, \tag{35}$$

where

$$(V_1 y)(t) = \left(V_{x^0 + \int_0^t y(s)\,ds}\right)(t)\ ,\ t \in R_+. \tag{36}$$

First order equations like (35) have been investigated in our paper [7], which allows to obtain in this fashion results for the second order equation (33).

It is useful to notice that after the substitution (34) in (33), the resulting equation (35) is also of causal type. See also our paper [6], which treats the linear case.

In the joint papers [4], [5], there are several results concerning existence of solutions for the problem (1), (2), regarding same types of asymptotic behavior. For instance, the boundedness of solutions has been emphasized. Also, the existence of solutions with (finite) limit at infinity.

This paper, as well as most of the papers cited in the list of references, are aimed at obtaining existence or behavior of solutions for functional equations involving general operators, not necessarily defined by classical operations (differentiation or integration). This trend has been observed in many publications during the last 4-5 decades, after the methods of Functional Analysis became common tools of investigation of functional equations.

References

[1] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

[2] C. Corduneanu, *Functional Equations with Causal Operators*, Taylor and Francis, 2002.

[3] C. Corduneanu, and M. Mahdavi, A class of second order functional differential equations of neutral type, *Mathematical Reports*, Romanian Academy, Vol. 5(55), No. 4, 2003, 293-299.

[4] C. Corduneanu, and M. Mahdavi, Neutral functional equations with causal operators on a semi-axis, *Nonlinear Dynamics and System Theory*, (to appear in 2009).

[5] C. Corduneanu, and M. Mahdavi, Neutral functional equations of the second order, *Functional Differential Equations*, (to appear).

[6] M. Mahdavi, Linear functional differential equations with abstract Volterra operators, *Differential and Integral Equations*, Vol.8, Number 6, July, 1995,1517,1523.

[7] M. Mahdavi, Asymptotic behavior in some classes of functional differential equations, *Nonlinear Dynamics and Systems Theory*, 4(1), 2004, 51-57.

[8] F. G. Tricomi, *Integral Equations*, Interscience Publishers, 1957.