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A System of Partial Differential Equations for Traffic Flow

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Abstract. A traffic flow model suggested by Lighthill, Whitham and Richards is discussed. A system of partial differential equations involving the density and velocity field for traffic flow is analyzed. The perturbation technique is used to linearize the system. A traveling wave solution of the corresponding linear or system is given.

1 Introduction

We discuss a hyperbolic system of partial differential equations with relaxation of the type introduced by Lighthill and Whitham [2] and Richards [3] to model traffic flow on long highways. Necessary definition and backgrounds can be found in many traffic flow theory articles and books. For example, Haberman [1, chapter 3] contains basic definitions and concepts.

Assume $\rho = \rho(x, t)$ denotes the density and $u = u(x, t)$ denotes the velocity field. Let $T(\rho)$ denote the reaction time. Assume $u = u(\rho)$. We discuss the following Cauchy problem.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + (u)\dfrac{\partial u}{\partial x} = -\dfrac{u - u_0}{T(\rho)}, \\ \rho(x,0) = f(x),\ u(x,0) = g(x). \end{cases} \tag{1}$$

In the above system, the first equation is called the continuity equation. The second equation contains the relaxation factor $T(\rho)$. In the second equation u_0 is an equilibrium speed. The last set of equations give initial density and velocity. Zingano [4] discusses nonlinear stability with decay rate for the traveling wave solution of the system above. After using perturbation technique, we obtain a traveling wave solution of the linearized system and discuss its stability.

In section 2, we use perturbation technique to linearize the system (1). We also use the change of variable along characteristic directions to obtain a simple linear system. In section 3, we solve the linear system and discuss stability of the solution.

2 Linearization

In this section we use perturbation technique to linearize the system (1). Linearization is useful for calibration and simulation of the model. It is also useful as an approximation to the model. Let ε be a small parameter. Assume ρ_0 and u_0 are equilibrium density and velocity, respectively. We define σ and ν by (2).

$$\begin{cases} \rho = \rho_0 + \varepsilon\sigma, \\ u = u_0 + \varepsilon\nu. \end{cases} \tag{2}$$

We substitute (2) in (1) and use a Taylor series expansion about (u_0, ρ_0). After using first order terms in terms of ε, we obtain a system of partial differential equations involving σ and ν.

The details are given below. We use the system (1) to obtain,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial x}(u) + \rho\dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + (u)\dfrac{\partial u}{\partial x} = -\dfrac{u-u_0}{T(\rho)}. \end{cases} \tag{3}$$

After substituting (2) in (3) and using of Taylor series expansion about (u_0, ρ_0) we obtain,

$$(\varepsilon)\frac{\partial \sigma}{\partial t} + (\varepsilon)\frac{\partial \sigma}{\partial x}(u_0 + \varepsilon\nu) + (\rho_0 + \varepsilon\sigma)(\varepsilon)\frac{\partial \nu}{\partial x} = 0$$

and

$$\begin{aligned} (\varepsilon)\frac{\partial \nu}{\partial t} + (u_0 + \varepsilon\nu)(\varepsilon)\frac{\partial \nu}{\partial x} &= -\frac{u-u_0}{T(\rho)}\left(1 + \frac{T'(\rho_0)}{T^2(\rho_0)}(\rho - \rho_0) + \dots\right) \\ &= -\frac{\varepsilon\nu}{T(\rho_0)}\left(1 + \frac{T'(\rho_0)}{T(\rho_0)^2}(\varepsilon\sigma) + \dots\right). \end{aligned}$$

By ignoring term of order ε^2 and division by ε we obtain,

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + (u_0)\dfrac{\partial \sigma}{\partial x} + (\rho_0)\dfrac{\partial \nu}{\partial x} = 0, \\ \dfrac{\partial \nu}{\partial t} + (u_0)\dfrac{\partial \nu}{\partial x} = -\dfrac{1}{T(\rho)}(\nu). \end{cases} \tag{4}$$

Appropriate initial conditions are $\sigma(x,0) = \dfrac{f(x) - \rho_0}{\varepsilon}$ and $\nu(x,0) = \dfrac{g(x) - u_0}{\varepsilon}$. System (4) is the linearization of system (1) which can be solved explicity. We use the change of variable $\xi = x - u_0 t$, $\tau = t$ to verify (4). By using the chain rule we obtain the system (5).

$$\begin{cases} \dfrac{\partial \sigma}{\partial \tau} + (\rho_0)\dfrac{\partial \nu}{\partial \xi} = 0, \\ \dfrac{\partial \nu}{\partial \tau} = -\dfrac{1}{T(\rho)}(\nu). \end{cases} \tag{5}$$

3 Approximate Solution

Let $T(\rho_0) = T_0$. In the system (5) the second equation can be solved by the separation of variables to obtain,

$$\nu = A(\xi)e^{-\tau/T_0} = A(x - u_0 t)e^{-t/T_0}.$$

Using the initial condition $\nu(x, 0) = \dfrac{g(x) - u_0}{\varepsilon}$, we conclude $A(x) = \dfrac{g(x) - u_0}{\varepsilon}$, thus

$$\nu(x, t) = \frac{g(x - u_0 t) - u_0}{\varepsilon} e^{-t/T_0}.$$

Using the relation $u = u_0 + \varepsilon\nu$ we obtain,

$$u(x, t) = u_0(1 - e^{-t/T_0}) + g(x - u_0 t)e^{-t/T_0}. \tag{6}$$

By substitution of ν in system (5) we obtain,

$$\frac{\partial\sigma}{\partial\tau} + \frac{\rho_0 g'(\xi)}{\varepsilon} e^{-\tau/T_0} = 0$$

Then we integrate to find

$$\sigma = -\frac{(\rho_0)g'(\xi)}{\varepsilon} \frac{e^{-\tau/T_0}}{(-1/T_0)} + h(\xi) = \frac{T_0\rho_0 g'(\xi)}{\varepsilon} e^{-\tau/T_0} + h(\xi).$$

Hence

$$\sigma(x, t) = \frac{T_0\rho_0 g'(x - u_0 t)}{\varepsilon} e^{-t/T_0} + h(x - u_0 t).$$

By using the initial condition,

$$\sigma(x, 0) = \frac{f(x) - \rho_0}{\varepsilon} = \frac{T_0\rho_0 g'(x)}{\varepsilon} + h(x)$$

Hence, $h(x) = \dfrac{f(x) - \rho_0 - T_0\rho_0 g'(x)}{\varepsilon}$. Now we obtain $\sigma(x, t)$ and thus we can find $\rho(x, t)$ approximately.

$$\sigma(x, t) = \frac{T_0\rho_0 g'(x - u_0 t)}{\varepsilon} e^{-t/T_0} + \frac{f(x - u_0 t) - \rho_0 - T_0\rho_0 g'(x - u_0 t)}{\varepsilon}$$

Then we use the approximation $f(x, t) = \rho_0 + \varepsilon\sigma(x, t)$ to find:

$$\rho(x, t) = T_0\rho_0 g'(x - u_0 t)(e^{-t/T_0} - 1) + f(x - u_0 t). \tag{7}$$

Equations (6) and (7) are approximate solutions for velocity and density. Note that as $t \to \infty$ then u approaches to u_0 and $\rho(x, t)$ approaches the traveling wave solution $f(x - u_0 t)$. Then the solution is asymptotically stable.

References

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