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On Certain Analytic Functions with Positive Real Part

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Abstract. The aim of the paper is to find certain conditions on the complex-valued functions $A, B : U \to \mathbb{C}$ defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ such that the differential inequality

$$\text{Re } [A(z)p^2(z)+B(z)p(z)+\alpha(zp'(z)-1)^3-3\beta\left(zp'(z)-\frac{b}{2}\right)^2+3\gamma(zp'(z))+\delta] > 0$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$, $b \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$.

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1 Introduction and preliminaries

We denote by $\mathcal{H}[U]$ the class of holomorphic functions in the open unit disc. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \quad f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [2, p.35].

Lemma 1.1 [2] *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ a function which satisfies*

$$\text{Re } \psi(\rho i, \sigma; z) \le 0, \tag{1.1}$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1+\rho^2)$, $z \in U$ and $n \ge 1$.

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } \psi\,(p(z), zp'(z); z) > 0 \tag{1.2}$$

then

$$\text{Re } p(z) > 0.$$

2 Main results

Following the work done in [1] we obtain the next theorem.

Theorem 2.1 *Let* $b \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\ \alpha \geq 0$, $\alpha + \beta \in \mathbb{R}_+$, $\alpha + \beta b + \gamma \in \mathbb{R}_+$,

$$\delta < \left(\frac{n^3}{8} + 1\right) \mathrm{Re}\ \alpha + \frac{3n^2}{4}(\alpha + \beta) + \frac{3n}{2}(\alpha + \beta b + \gamma) + \frac{3b^2}{4}\mathrm{Re}\ \beta$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i)\ \mathrm{Re}\ A(z) > -\frac{3n^3}{8}\mathrm{Re}\ \alpha - \frac{3n^2}{2}(\alpha + \beta) - \frac{3n}{2}(\alpha + \beta b + \gamma);$$

$$(ii)\ \mathrm{Im}^2 B(z) \leq 4\left[\frac{3n^3}{8}\mathrm{Re}\ \alpha + \frac{3n^2}{2}(\alpha+\beta) + \frac{3n}{2}(\alpha+\beta b+\gamma) + \mathrm{Re}\ A(z)\right] \cdot \tag{2.1}$$

$$\cdot\left[\left(\frac{n^3}{8}+1\right)\mathrm{Re}\ \alpha + \frac{3n^2}{4}(\alpha + \beta) + \frac{3n}{2}(\alpha + \beta b + \gamma) + \frac{3b^2}{4}\mathrm{Re}\ \beta - \delta\right].$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\mathrm{Re}\ [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - 1)^3 - 3\beta\left(zp'(z) - \frac{b}{2}\right)^2 + 3\gamma(zp'(z)) + \delta] > 0 \tag{2.2}$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) + B(z)p(z) + \tag{2.3}$$

$$+\alpha(zp'(z) - 1)^3 - 3\beta\left(zp'(z) - \frac{b}{2}\right)^2 + 3\gamma(zp'(z)) + \delta].$$

From (2.2) we get

$$\mathrm{Re}\ \psi(p(z), zp'(z); z) > 0, \quad z \in U. \tag{2.4}$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1 + \rho^2)$, hence

$$-\sigma^2 \leq -\frac{n^2}{4}(1 + \rho^2)^2,\ \sigma^3 \leq -\frac{n^3}{8}(1 + \rho^2)^3$$

and $z \in U$, by using (2.1) we obtain

$$\mathrm{Re}\ \psi(\rho i, \sigma; z) =$$

$$= \mathrm{Re}\ [A(z)(\rho i)^2 + B(z)\rho i + \alpha(\sigma - 1)^3 - 3\beta\left(\sigma - \frac{b}{2}\right)^2 + 3\gamma\sigma + \delta] =$$

$$-\rho^2\text{Re } A(z) - \rho\text{Im } B(z) + (\sigma^3 - 1)\text{Re } \alpha - 3(\alpha + \beta)\sigma^2 +$$
$$+3\left(\alpha + \beta b + \gamma\right)\sigma - \frac{3b^2}{4}\text{Re } \beta + \delta \leq$$
$$-\rho^2\text{Re } A(z) - \rho\text{Im } B(z) - \frac{n^3}{8}(1+\rho^2)^3\text{Re } \alpha - \text{Re } \alpha - \frac{3n^2}{4}(\alpha+\beta)(1+\rho^2)^2 -$$
$$-\frac{3n}{2}\left(\alpha + \beta b + \gamma\right)(1+\rho^2) - \frac{3b^2}{4}\text{Re } \beta + \delta =$$
$$-\frac{n^3}{8}\rho^6\text{Re } \alpha - \left[\frac{3n^3}{8}\text{Re } \alpha + \frac{3n^2}{4}(\alpha + \beta)\right]\rho^4 -$$
$$-\left[\left(\frac{3n^3}{8}\text{Re } \alpha + \frac{3n^2}{2}(\alpha + \beta) + \frac{3n}{2}\left(\alpha + \beta b + \gamma\right) + \text{Re } A(z)\right)\rho^2 +\right.$$
$$+\rho\text{Im } B(z) + \left(\frac{n^3}{8} + 1\right)\text{Re } \alpha + \frac{3n^2}{4}\left(\alpha + \beta\right) +$$
$$\left. + \frac{3n}{2}\left(\alpha + \beta b + \gamma\right) + \frac{3b^2}{4}\text{Re } \beta - \delta\right] \leq 0.$$

By using Lemma 1.1 we have Re $p(z) > 0$. □

Taking $\beta = \gamma = \bar{\alpha}$ in the Theorem 2.1, we have

Corollary 2.1 *Let* $b \in \mathbb{R}_+$, $\alpha \in \mathbb{C}$, Re $\alpha \geq 0$,

$$\delta < \left(\frac{n^3}{8} + \frac{3n^2}{2} + \frac{3n}{2}(2+b) + \frac{3b^2}{4} + 1\right) \cdot \text{Re } \alpha$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned}
&(i)\ \text{Re } A(z) > \left[-\frac{3n^3}{8} - 3n^2 - \frac{3n}{2}(2+b)\right]\text{Re } \alpha;\\
&(ii)\ \text{Im}^2 B(z) \leq 4 \cdot \left[\left(\frac{3n^3}{8} + 3n^2 - \frac{3n}{2}(2+b)\right) \cdot \text{Re } \alpha + \text{Re } A(z)\right] \cdot\\
&\qquad \cdot \left[\left(\frac{n^3}{8} + \frac{3n^2}{2} + \frac{3n}{2}(2+b) + \frac{3b^2}{4} + 1\right)\text{Re } \alpha - \delta\right].
\end{aligned} \tag{2.5}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\text{Re } \left[A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - 1)^3 - 3\bar{\alpha}\left(zp'(z) - \frac{b}{2}\right)^2 + \right. \tag{2.6}$$
$$\left. +3\bar{\alpha}(zp'(z)) + \delta\right] > 0$$

then

$$\text{Re } p(z) > 0.$$

Taking $\alpha+\beta=\alpha+\beta b+\gamma=\alpha+\gamma=1$ in the Theorem 2.1, we obtain

Corollary 2.2 *Let* $b\in\mathbb{R}_+$, $\alpha\in\mathbb{C}$, $\mathrm{Re}\ \alpha\geq 0$,

$$\delta<\left(\frac{n^3}{8}+1\right)\mathrm{Re}\ \alpha+\frac{3n^2}{4}+\frac{3n}{2}+\frac{3b^2}{4}(1-\alpha)$$

and n *be a positive integer. Suppose that the functions* $A,B:U\to\mathbb{C}$ *satisfy*

$$\begin{aligned}&(i)\ \mathrm{Re}\ A(z)>-\frac{3n^3}{8}\mathrm{Re}\ \alpha-\frac{3n^2}{2}-\frac{3n}{2};\\&(ii)\ \mathrm{Im}\ ^2B(z)\leq\ 4\cdot\left[\frac{3n^3}{8}\mathrm{Re}\ \alpha+\frac{3n^2}{2}+\frac{3n}{2}+\mathrm{Re}\ A(z)\right]\cdot\\&\qquad\cdot\left[\left(\frac{n^3}{8}+1\right)\mathrm{Re}\ \alpha+\frac{3n^2}{4}+\frac{3n}{2}+\frac{3b^2}{4}(1-\alpha)-\delta\right].\end{aligned}\tag{2.7}$$

If $p\in\mathcal{H}[1,n]$ *and*

$$\mathrm{Re}\ [A(z)p^2(z)+B(z)p(z)+\alpha(zp'(z)-1)^3-3(1-\alpha)\left(zp'(z)-\frac{b}{2}\right)^2+\tag{2.8}$$

$$+3(1-\alpha)(zp'(z))+\delta]>0$$

then

$$\mathrm{Re}\ p(z)>0.$$

Taking $\alpha=0$ in the Theorem 2.1, we have

Corollary 2.3 *Let* $b\in\mathbb{R}_+$, $\beta,\gamma>0$, $\delta<\frac{3n^2}{4}\beta+\frac{3n}{2}(\beta b+\gamma)+\frac{3n^2}{4}$ *and* n *be a positive integer. Suppose that the functions* $A,B:U\to\mathbb{C}$ *satisfy*

$$\begin{aligned}&(i)\ \mathrm{Re}\ A(z)>-\frac{3n^2}{2}\beta-\frac{3n}{2}(\beta b+\gamma);\\&(ii)\ \mathrm{Im}\ ^2B(z)\leq 4\left(\frac{3n^2}{2}\beta+\frac{3n}{2}(\beta b+\gamma)+\mathrm{Re}\ A(z)\right)\left(\frac{3n^2}{4}\beta+\frac{3n}{2}(\beta b+\gamma)-\delta\right).\end{aligned}\tag{2.9}$$

If $p\in\mathcal{H}[1,n]$ *and*

$$\mathrm{Re}\,[A(z)p^2(z)+B(z)p(z)-3\beta\left(zp'(z)-\frac{b}{2}\right)^2+3\gamma(zp'(z))+\delta]>0\tag{2.10}$$

then

$$\mathrm{Re}\ p(z)>0.$$

Taking $\beta=\gamma=0$ in the Theorem 2.1, we obtain

Corollary 2.4 *Let* $b \in \mathbb{R}_+$, $\alpha > 0$, $\delta < \left(\frac{n^3}{8} + \frac{3n^2}{4} + \frac{3n}{2} + 1\right) \cdot \alpha$ *and* n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &(i)\ \mathrm{Re}\ A(z) > -\left(\frac{3n^3}{8} - \frac{3n^2}{2} - \frac{3n}{2}\right) \cdot \alpha; \\ &(ii)\ \mathrm{Im}\ ^2 B(z) \le 4\left[\left(\frac{3n^3}{8} + \frac{3n^2}{2} + \frac{3n}{2}\right)\alpha + \mathrm{Re}\ A(z)\right] \cdot \\ &\qquad \cdot \left[\left(\frac{n^3}{8} + \frac{3n^2}{4} + \frac{3n}{2} + 1\right)\alpha - \delta\right]. \end{aligned} \tag{2.11}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$\mathrm{Re}\ [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - 1)^3 + \delta] > 0 \tag{2.12}$$

then

$$\mathrm{Re}\ p(z) > 0.$$

Letting $n = 1$, $b = 1$, $\alpha = 2 + i$, $\delta = 15$, $A(z) = 1 - z$ and $B(z) = 1 + 2z$ in Corollary 2.1, we have

Example 2.1 *If* $p \in \mathcal{H}[1, 1]$ *and*

$$\mathrm{Re}\,[(1-z)p^2(z) + (1+2z)p(z) + (2+i)(zp'(z)-1)^3 - 3(2-i)\left(zp'(z) - \frac{1}{2}\right)^2 + \tag{2.13}$$

$$+3(2-i)(zp'(z)) + \delta] > 0$$

then

$$\mathrm{Re}\ p(z) > 0.$$

References

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