LIBERTAS MATHEMATICA, vol XXIX (2009)

An Extended Hardy-Hilbert Integral Inequality

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Abstract. In this paper, it is shown that an extended Hardy-Hilbert integral inequality with a parameter can be established by introducing two nonnegative functions $u(x)$and $v(x)$which are differentiable in interval $(0, +\infty)$, and the constant factor is proved to be the best possible. In particular, for case $p = 2$, some extensions of the classical Hilbert integral inequality are built. As applications, some important inequalities are studied, when $u(x)$and $v(x)$are power function, exponent function and logarithm function, and some equivalent forms are considered.

Keywords: Hardy-Hilbert's integral inequality, weight function, monotonous function, beta function, gamma function.

Mathematics Subject Classification (2000): 26D15, 33B15.

1 Introduction

Let $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, $f(x), g(x) \geq 0$, $f(x) \in L^p(0, +\infty)$ and $g(x) \in L^q(0, +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)\,dy\right\}^{\frac{1}{q}} \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin \pi/p}$ is the best possible. And the equality in (1.1) holds if and only if $f(x) = 0$ or $g(x) = 0$. This is famous Hardy-Hilbert's integral inequality (see [1]). Recently, the inequality (1.1) was extended in some papers (such as [2]–[7] etc.). The purpose of the present paper is to establish the following inequality of the form

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(u(x) + v(y))^\lambda}dxdy \leq k(\lambda) \left\{\int_0^\infty \omega_q(\lambda,\ x)\,f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \omega_p(\lambda,\ x)\,g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{1.2}$$

where $1 - \frac{q}{p} < \lambda \leq 2$, and to decide the coefficient $k(\lambda)$ and to prove $k(\lambda)$to be the best possible, at same time to find the expression of the weight function $\omega_r(\lambda,\ x)$ $(r = p, q)$. And the various results appeared in some papers (such as [2]-[7] etc.) are merely the particular cases of this paper. For convenience, the beta function $B(\lambda - \frac{2-\lambda}{p},\ \frac{2-\lambda}{p})$is denoted by $\tilde{B}$. Throughout this paper we will frequently use this notation. .

2 Some Lemmas

In order to prove our assertions we need the following lemmas.

Lemma 2.1. *Let $r > 1$, $0 \le rs < 1$ and $\lambda > 1 - rs$. Then*

$$\int_0^\infty \frac{1}{(1\ +\ t\ )^\lambda} \left(\frac{1}{t}\right)^{rs} dt = B\left(\lambda - (1 - rs),\ 1 - rs\right)\ , \tag{2.1}$$

where $B(p, q)$ is the beta function.

Lemma 2.2. *Let $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, $0 \le ps < 1$ and $1 - qs < \lambda \le 2$. Define a function Φ by*

$$\Phi(s) = \left\{B\left(\lambda - (1 - ps)\,,\ 1 - ps\right)\right\}^{\frac{1}{p}} \left\{B\left(\lambda - (1 - qs)\,,\ 1 - qs\right)\right\}^{\frac{1}{q}} \tag{2.2}$$

where $B(m, n)$ is the beta function. Then $\Phi(s)$ attains the minimum $\tilde{B}$, when $s = \frac{2-\lambda}{pq}$.

Proof. Based on the relation $B\left(m, n\right) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m\ +\ n)}$, where $\Gamma\left(z\right)$ is the gamma function, we can write (2.2) as

$$\Phi\left(s\right) = \frac{1}{\Gamma(\lambda)}\left(I_p^{\frac{1}{p}} I_q^{\frac{1}{q}}\right),$$

where $I_r = \Gamma\left(1 - rs\right)\Gamma\left(\lambda - (1 - rs)\right), \quad r = p, q.$

Taking the derivative of $\Phi\left(s\right)$ we have $\Phi'\left(s\right) = \Phi\left(s\right)\Psi\left(s\right)$, where

$$\Psi\left(s\right) = -\psi\left(1 - ps\right) + \psi\left(\lambda - (1 - ps)\right) - \psi\left(1 - qs\right) + \psi\left(\lambda - (1 - qs)\right),$$

here $\psi\left(z\right) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the psi function. We choose thus s such that $1 - ps = \lambda - (1 - qs)$, so that $1 - qs = \lambda - (1 - ps)$, hence $s = \frac{2-\lambda}{p+q}$. Since that $\frac{1}{p}\ +\ \frac{1}{q} = 1$, it follows that $s = \frac{2-\lambda}{pq}$. We therefore have $\Psi\left(\frac{2-\lambda}{pq}\right) = 0$. i.e. $\Phi'\left(\frac{2-\lambda}{pq}\right) = 0$. It is known from the paper [7] that $\psi'\left(z\right) = \zeta\left(2, z\right) = \sum\limits_{n=1}^{\infty} \frac{1}{(z\ +\ n)^2}$, where ζ is the Riemann zeta function. It follows that $\Psi'(s)0$, hence $\Psi(s)$ is strictly increasing. Owing to the fact that $\Psi\left(\frac{2-\lambda}{pq}\right) = 0$, $\Psi\left(s\right) > 0$ when $s > \frac{2-\lambda}{pq}$. This shows that $\Phi'\left(s\right) > 0$. Similarly, we have $\Phi'\left(s\right) < 0$ when $s < \frac{2-\lambda}{pq}$. Consequently, the minimum of $\Phi\left(s\right)$ is that

$$\Phi\left(\frac{2-\lambda}{pq}\right) = \left(B\left(\lambda - \left(1 - \frac{2-\lambda}{q}\right), 1 - \frac{2-\lambda}{q}\right)\right)^{\frac{1}{p}} \left(B\left(\lambda - \left(1 - \frac{2-\lambda}{p}\right), 1 - \frac{2-\lambda}{p}\right)\right)^{\frac{1}{q}}.$$

Since $1 - \frac{2-\lambda}{q} = \lambda - \left(1 - \frac{2-\lambda}{p}\right)$, $1 - \frac{2-\lambda}{p} = \lambda - \left(1 - \frac{2-\lambda}{q}\right)$ and $B\left(m, n\right) = B\left(n, m\right)$, we have the relation:

$$B\left(\lambda - \left(1 - \frac{2-\lambda}{q}\right),\ 1 - \frac{2-\lambda}{q}\right) = B\left(\lambda - \left(1 - \frac{2-\lambda}{p}\right),\ 1 - \frac{2-\lambda}{p}\right).$$

We therefore obtain $\Phi\left(\frac{2-\lambda}{pq}\right) = \tilde{B}$. The Lemma is proved.

3 Main Results

In the section, we will apply the above lemmas to build some new inequalities on Hardy-Hilbert's integral type. Let the nonnegative functions $u(x)$and $v(x)$are differentiable in $(0, +\infty)$, $u'(x) > 0$ and $v'(x) > 0$, $u(\infty) = v(\infty) = \infty$ and $u(1) = v(1)$. Then we have the following result.

Theorem 3.1. *Let* $f(x), g(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$, $1 - \frac{q}{p} < \lambda \leq 2$. *If* $\int_0^\infty \left\{(u(x))^{1-\lambda}(u'(x))^{1-p}\right\} f^p(x)\,dx < +\infty$ *and* $\int_0^\infty \left\{(v(x))^{1-\lambda}(v'(x))^{1-q}\right\} g^q(x)\,dx < +\infty$, *then*

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy &\leq \tilde{B}\left\{\int_0^\infty \left\{(u(x))^{1-\lambda}(u'(x))^{1-p}\right\} f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\times \left\{\int_0^\infty \left\{(v(x))^{1-\lambda}(v'(x))^{1-q}\right\} g^q(x)\,dx\right\}^{\frac{1}{q}}.\end{aligned} \tag{3.1}$$

And the equality in (3.1) holds if and only if$f(x) = 0$ or $g(x) = 0$. If the constant factors of the functions $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$ are not considered, then the coefficient $\tilde{B}$ is the best possible.

Proof. Let $s > p$, $f(x) = F(x)\{u'(x)\}^{\frac{1}{q}}$and $g(y) = G(y)\{v'(y)\}^{\frac{1}{p}}$. Define two functions by

$$\alpha = \left(\frac{F(x)\{v'(y)\}}{(u(x)+v(y))^\lambda}\right)^{\frac{1}{p}}\left(\frac{u(x)}{v(y)}\right)^s \text{ and } \beta = \left(\frac{G(y)\{u'(x)\}}{(u(x)+v(y))^\lambda}\right)^{\frac{1}{q}}\left(\frac{v(y)}{u(x)}\right)^s \tag{3.2}$$

Let's apply Hölder's inequality to estimate the left hand side of (3.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy = \int_0^\infty\int_0^\infty \alpha\beta\,dxdy \leq \left\{\int_0^\infty\int_0^\infty \alpha^p dxdy\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty \beta^q dxdy\right\}^{\frac{1}{q}}. \tag{3.3}$$

It is easy to deduce that

$$\int_0^\infty\int_0^\infty \alpha^p dxdy = \int_0^\infty\int_0^\infty \frac{F^p(x)\{v'(y)\}}{(u(x)+v(y))^\lambda}\left(\frac{u(x)}{v(y)}\right)^{ps} dxdy = \int_0^\infty \varpi(p, \lambda, x)\,F^p(x)\,dx.$$

Based on Lemma 2.1 we compute the weight function ϖ as follows:

$$\varpi(p, \lambda, x) = \int_0^\infty \frac{v'(y)}{(u(x)+v(y))^\lambda}\left(\frac{u(x)}{v(y)}\right)^{ps} dy = \int_{v(0)/u(x)}^\infty \frac{(u(x))^{1-\lambda}}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{ps} dt$$

$$\leq \int_0^\infty \frac{(u(x))^{1-\lambda}}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{ps} dt = \left\{(u(x))^{1-\lambda} B(\lambda - (1-ps), 1-ps)\right\}.$$

Notice that $F(x) = \{u'(x)\}^{-\frac{1}{q}} f(x)$, hence we have

$$\int_0^\infty \int_0^\infty \alpha^p dxdy \le B(\lambda - (1-ps),\ 1-ps) \int_0^\infty (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x)\, dx \tag{3.4}$$

Similarly, we have

$$\int_0^\infty \int_0^\infty \beta^q dxdy \le B(\lambda - (1-qs),\ 1-qs) \int_0^\infty (v(x))^{1-\lambda} (v'(x))^{1-q} g^q(x)\, dx \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda} dxdy \le\ & \Phi(s) \left\{ \int_0^\infty (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x)\, dx \right\}^{\frac{1}{p}} \\ & \times \left\{ (v(x))^{1-\lambda} (v'(x))^{1-q} \int_0^\infty g^q(x)\, dx \right\}^{\frac{1}{q}} \end{aligned} \tag{3.6}$$

where $\Phi(s)$ is defined by (2.2).

It follows from Lemma 2.2 that the minimum of $\Phi(s)$ is $\tilde{B}$, where λ satisfies the constraint $1 - \frac{q}{p} < \lambda \le 2$. As a result, it follows from (3.6) that the inequality (3.1) is yielded at once. And it is obvious that the equality in (3.1) holds if and only if $f(x) = 0$ or $g(x) = 0$.

It remains to need only to show that $\tilde{B}$ is the best possible. Define two functions by

$$\tilde{f}(x) = \begin{cases} 0 & x \in (0,\ 1) \\ (u(x))^{-\frac{2-\lambda+\varepsilon}{p}} (u'(x)) & x \in [1,\ +\infty) \end{cases}$$

and

$$\tilde{g}(y) = \begin{cases} 0 & y \in (0,\ 1) \\ (v(y))^{-\frac{2-\lambda+\varepsilon}{q}} (v'(y)) & y \in [1,\ +\infty) \end{cases}$$

Let $u(1) = v(1) = \beta$. It is easy to deduce that

$$\int_0^{+\infty} (u(x))^{1-\lambda} (u'(x))^{1-p} \tilde{f}^p(x)\, dx = \int_1^{+\infty} (u(x))^{-1-\varepsilon} du = \frac{1}{\varepsilon\beta^\varepsilon}\ .$$

Similarly, we have

$$\int_0^\infty (v(y))^{1-\lambda} (v'(y))^{1-q} \tilde{g}^q(y)\, dy = \frac{1}{\varepsilon\beta^\varepsilon}.$$

If $\tilde{B}$ is not best possible, then there exists $C > 0$ and $C < \tilde{B}$ such that

$$\int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(u(x)+v(y))^\lambda} dxdy \le C \left(\int_0^\infty \omega_p \tilde{f}^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty \omega_q \tilde{g}^q(x)\, dx \right)^{\frac{1}{q}} = \frac{C}{\varepsilon\beta^\varepsilon} \tag{3.7}$$

where $\omega_p = (u(x))^{1-\lambda}(u'(x))^{1-p}$and $\omega_q = (v(x))^{1-\lambda}(v'(x))^{1-q}$.

On the other hand, we have

$$\int_0^\infty\int_0^\infty \frac{\tilde f(x)\tilde g(y)}{(u(x)+v(y))^\lambda}dxdy \int_1^\infty\int_1^\infty \frac{\left\{(u(x))^{-\frac{2-\lambda+\varepsilon}{p}}\left(u'(x)\right)\right\}\left\{(v(y))^{-\frac{2-\lambda+\varepsilon}{q}}\left(v'(y)\right)\right\}}{(u(x)+v(y))^\lambda}dxdy$$

$$= \int_1^\infty\left\{\int_1^\infty \frac{(v(y))^{-\frac{2-\lambda+\varepsilon}{q}}\left(v'(y)\right)}{(u(x)+v(y))^\lambda}dy\right\}\left\{(u(x))^{-\frac{2-\lambda+\varepsilon}{p}}\left(u'(x)\right)\right\}dx$$

$$= \int_1^\infty\left\{\int_{v(1)/u(x)}^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt\right\}\left\{(u(x))^{-1-\varepsilon}\left(u'(x)\right)\right\}dx$$

$$= \int_1^\infty\left\{\int_{v(1)/u(x)}^1\left\{(u(x))^{-1-\varepsilon}\left(u'(x)\right)\right\}dx\right\}\frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt$$
$$+\int_1^\infty\left\{\int_1^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt\right\}\left\{(u(x))^{-1-\varepsilon}\left(u'(x)\right)\right\}dx$$

$$= \int_0^1\left\{\int_{v(1)/t}^\infty (u(x))^{-1-\varepsilon}d(u(x))\right\}\frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt$$
$$+\frac{1}{\varepsilon\beta^\varepsilon}\int_1^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt$$

$$= \frac{1}{\varepsilon\beta^\varepsilon}\int_0^1 \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda-(q-1)\varepsilon}{q}}dt + \frac{1}{\varepsilon\beta^\varepsilon}\int_1^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt \tag{3.8}$$

When ε is sufficiently small, by Lemma 1.1, we have

$$= \int_0^1 \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda-(q-1)\varepsilon}{q}}dt + \int_1^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda+\varepsilon}{q}}dt$$

$$= \left(\int_0^1 \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda}{q}}dt + \circ_1(1)\right) + \left(\int_1^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda}{q}}dt + \circ_2(1)\right)$$

$$= \int_0^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{\frac{2-\lambda}{q}}dt + \circ(1) = \tilde B + \circ(1) \qquad (\varepsilon\to 0) \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\int_0^\infty\int_0^\infty \frac{\tilde f(x)\tilde g(y)}{(u(x)+v(y))^\lambda}dxdy = \frac{1}{\varepsilon\beta^\varepsilon}\left\{\tilde B + \circ(1)\right\}. \quad (\varepsilon\to 0) \tag{3.10}$$

Clearly, when ε is sufficiently small, the inequality (3.7) is in contradiction with (3.10). Therefore, $\tilde{B}$ in (3.1) is the best possible. The proof of Theorem is completed.

When $p = 2$, $\tilde{B}$ is reduced to $B\left(\frac{\lambda}{2},\ \frac{\lambda}{2}\right)$. Hence we have the following result.

Corollary 3.2. With the assumptions as Theorem 3.1, if

$$\int_0^\infty \left\{(u(x))^{1-\lambda}(u'(x))^{-1}\right\} f^2(x)\,dx < +\infty \text{ and } \int_0^\infty \left\{(v(x))^{1-\lambda}(v'(x))^{-1}\right\} g^2(x)\,dx < +\infty$$

then

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy \ \le\ & B\left(\tfrac{\lambda}{2},\ \tfrac{\lambda}{2}\right)\left\{\int_0^\infty \left\{(u(x))^{1-\lambda}(u'(x))^{-1}\right\} f^2(x)\,dx\right\}^{\frac{1}{2}} \\ & \times\left\{\int_0^\infty \left\{(v(x))^{1-\lambda}(v'(x))^{-1}\right\} g^2(x)\,dx\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

And the equality in (3.11) holds if and only if $f(x) = 0$ or $g(x) = 0$. If the constant factors of the functions $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$ are not considered, then the coefficient $B\left(\frac{\lambda}{2},\ \frac{\lambda}{2}\right)$is best possible.

The inequality (3.11) is obviously an extension of the results of the papers [3] and [4].

In particular, for case $\lambda = 1$, $\tilde{B}$ can be reduced to $\frac{\pi}{\sin\ \pi/p}$. Hence an extension of (1.1) is attained.

Corollary 3.3. With the assumptions as Theorem 3.1, if $\int_0^\infty (u'(x))^{1-p} f^p(x) < +\infty$ and $\int_0^\infty (v'(x))^{1-q} g^q(x)\,dx < +\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{u(x)+v(y)}dxdy \le \frac{\pi}{\sin\ \pi/p}\left\{\int_0^\infty (u'(x))^{1-p} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty (v'(x))^{1-q} g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{3.12}$$

where the constant factor $\frac{\pi}{\sin\ \pi/p}$ is the best possible.

When $p = 2$ Hilbert's integral type inequality with weights can be built.

Corollary 3.4. If $\int_0^\infty (u'(x))^{-1} f^2(x)\,dx < +\infty$ and $\int_0^\infty (v'(x))^{-1} g^2(x)\,dx < +\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{u(x)+v(y)}dxdy \le \pi\left\{\int_0^\infty (u'(x))^{-1} f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty (v'(x))^{-1} g^2(x)\,dx\right\}^{\frac{1}{2}}$$

where the constant factor π is the best possible.

Now, we consider the constant factors of the functions $u(x)$, $v(x)$, $u'(x)$and $v'(x)$.

Let the constant factors of $u(x)$, $v(x)$, $u'(x)$ and $v'(x)$are in turn C_1, C_2, C_3, C_4. Then they can be written in form:

$$u(x) = C_1\tilde{u}(x), \quad v(x) = C_2\tilde{v}(x), \quad u'(x) = C_3\tilde{u}'(x) \text{ and } v'(x) = C_4\tilde{v}'(x) \tag{3.13}$$

And then suppose that

$$a = \left(\frac{C_1^{1-\lambda}}{C_4}\right)^{\frac{1}{p}} \left(\frac{C_2^{1-\lambda}}{C_3}\right)^{\frac{1}{q}} \tag{3.14}$$

Then we have the following important result:

Theorem 3.5. *With the assumptions as Theorem 3.1, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(u(x)+v(y))^\lambda}dxdy \le a\tilde{B}\left\{\int_0^\infty \left\{(\tilde{u}(x))^{1-\lambda}\left(\tilde{u'}(x)\right)^{1-p}\right\} f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_0^\infty \left\{(\tilde{v}(v))^{1-\lambda}\left(\tilde{v'}(x)\right)^{1-q}\right\} g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{3.15}$$

where the constant factor $a\tilde{B}$ is the best possible and the constant a is given by (3.14), $\tilde{u}(x)$ and $\tilde{v}(x)$as well as their derivatives are given by (3.13).

For case $\lambda = 1$, from the theorem 3.5 we obtain the following

Corollary 3.6. With the assumptions as Theorem 3.1, if $\int_0^\infty (u'(x))^{1-p} f^p(x) < +\infty$ and $\int_0^\infty (v'(x))^{1-q} g^q(x)\,dx < +\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{u(x)+v(y)}dxdy \le \frac{b\pi}{\sin\,\pi/p}\left\{\int_0^\infty \left(\tilde{u'}(x)\right)^{1-p} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \left(\tilde{v'}(x)\right)^{1-q} g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{3.16}$$

where the constant factor $\frac{b\pi}{\sin\,\pi/p}$ is the best possible and the constant $b = C_4^{-\frac{1}{p}}C_3^{-\frac{1}{q}}$, $\tilde{u}(x)$ and $\tilde{v}(x)$as well as their derivatives are given by (3.13).

Corollary 3.7. If $\int_0^\infty (u'(x))^{-1} f^2(x)\,dx < +\infty$ and $\int_0^\infty (v'(x))^{-1} g^2(x)\,dx < +\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{u(x)+v(y)}dxdy \le c\pi\left\{\int_0^\infty \left(\tilde{u'}(x)\right)^{-1} f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \left(\tilde{v'}(x)\right)^{-1} g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{3.17}$$

where the constant factor $c\pi$ is the best possible and the constant $c = \frac{1}{\sqrt{C_3C_4}}$, $\tilde{u}(x)$ and $\tilde{v}(x)$as well as their derivatives are given by (3.13).

4 Some Applications

In this section we enumerate only the cases which $u(x)$ and $v(x)$are power function, exponent function and logarithm function.

1) **Power function.**

Let $u(x) = x^{\alpha}$ ($\alpha > 0$) and $v(y) = y^{\beta}$ ($\beta > 0$). In according to (3.14), it is easy to deduce that $a = \alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}$. By Theorem 3.5, we have the following result:

Theorem 4.1. *Let* $f(x), g(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$, $1 - \frac{q}{p} < \lambda \leq 2$. *If*

$$\int_0^{\infty} \left(x^{(1-\lambda)\alpha-(\alpha-1)(p-1)}\right) f^p(x)\,dx < +\infty \text{ and } \int_0^{\infty} \left(x^{(1-\lambda)\beta-(\beta-1)(q-1)}\right) g^q(x)\,dx < +\infty,$$

then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x^{\alpha} + y^{\beta})^{\lambda}}dxdy \leq a\tilde{B}\left\{\int_0^{\infty}\left\{x^{(1-\lambda)\alpha-(\alpha-1)(p-1)}\right\} f^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_0^{\infty}\left\{x^{(1-\lambda)\beta-(\beta-1)(q-1)}\right\} g^q(x)\,dx\right\}^{\frac{1}{q}} \tag{4.1}$$

where the constant factor $a\tilde{B}$*is the best possible, and* $a = \alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}$.

In particular, if $\alpha = \beta = 2$, $\lambda = 1$ and $p = 2$, then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^2 + y^2}dxdy \leq \frac{\pi}{2}\left\{\int_0^{\infty} x^{-1}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^{\infty} x^{-1}g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{4.2}$$

where the constant factor $\frac{\pi}{2}$ is the best possible.

2) **Exponent function.**

Let $u(x) = e^x$ and $v(y) = e^y$. It is known from (3.14) that $a = 1$. In according to Theorem 3.5, we have the following result:

Theorem 4.2. *Let* $f(x), g(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$ *and* $1 - \frac{q}{p} < \lambda \leq 2$. *If* $\int_0^{\infty} e^{x(2-\lambda-p)} f^p(x)\,dx < +\infty$ *and* $\int_0^{\infty} e^{x(2-\lambda-q)} g^q(x)\,dx < +\infty$, *then*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(e^x + e^y)^{\lambda}}dxdy \leq \tilde{B}\left\{\int_0^{\infty} e^{x(2-\lambda-p)} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty} e^{x(2-\lambda-q)} g^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{4.3}$$

where the constant factor $\tilde{B}$ *is the best possible.*

In particular, if $\lambda = 1$ and $p = 2$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{e^x + e^y} dxdy \leq \pi \left\{ \int_0^\infty e^{-x} f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty e^{-x} g^2(x)\, dx \right\}^{\frac{1}{2}}, \tag{4.4}$$

where the constant factor π is the best possible.

3) **Logarithm function.**

Let $u(x) = \ln(1+x)$ and $v(y) = \ln(1+y)$. Then it is known from (3.14) that $a = 1$. In according to Theorem 3.1, we get an extension on the corresponding result of the paper [8].

Theorem 4.3. *Let $f(x), g(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$, $1 - \frac{q}{p} < \lambda \leq 2$. If $\int_0^\infty (\ln(1+x))^{1-\lambda} (1+x)^{p-1} f^p(x)\, dx < +\infty$ and $\int_0^\infty (\ln(1+x))^{1-\lambda} (1+x)^{q-1} g^q(x)\, dx < +\infty$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\ln(1+x) + \ln(1+y))^\lambda} dxdy \leq \tilde{B} \left\{ \int_0^\infty (\ln(1+x))^{1-\lambda} (1+x)^{p-1} f^p(x)\, dx \right\}^{1/p}$$

$$\times \left\{ \int_0^\infty (\ln(1+x))^{1-\lambda} (1+x)^{q-1} g^q(x)\, dx \right\}^{1/q}, \tag{4.5}$$

where the constant factor $\tilde{B}$ is the best possible.

In particular, if $p = 2$, or $p = 2$ and $\lambda = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(1+x) + \ln(1+y)} dxdy \leq \pi \left\{ \int_0^\infty (1+x) f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty (1+x) g^2(x)\, dx \right\}^{\frac{1}{2}} \tag{4.6}$$

where the constant factor π is the best possible.

Based on (3.1) and (3.15), a great deal of new inequalities might be established. Here they are omitted.

5 Some Equivalent Forms

In this section we will build some equivalent inequalities each other.

Theorem 5.1. *Let $f(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$, $1 - \frac{q}{p} < \lambda \leq 2$. If $\int_0^\infty \left\{ (u(x))^{1-\lambda} (u'(x))^{1-p} \right\} f^p(x)\, dx < +\infty$, then*

$$\int_0^\infty \left\{ (v(y))^{\lambda-1} (v'(y))^{q-1} \right\}^{p-1} \left(\int_0^\infty \frac{f(x)}{(u(x)+v(y))^\lambda} dx \right)^p dy$$

$$\leq \tilde{B}^{p}\int_{0}^{\infty}\left\{(u(x))^{1-\lambda}(u'(x))^{1-p}\right\}f^{p}(x)\,dx. \tag{5.1}$$

If the constant factors of the functions $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$ are not considered, then the coefficient $\tilde{B}^{p}$ in (5.1) is the best possible. And the inequality (5.1) is equivalent to (3.1).

Proof. First, we suppose that the inequality (3.1) is valid. Setting a real function $g(y)$ as

$$g(y)=\left\{\left(v(y)^{\lambda-1}v'(y)^{q-1}\right)\int_{0}^{\infty}\frac{1}{(u(x)+v(y))^{\lambda}}f(x)\,dx\right\}^{p-1}. \quad y\in(0,\,+\infty)$$

We have

$$\int_{0}^{\infty}\left((v(y))^{\lambda-1}(v'(y))^{q-1}\right)^{p-1}\left\{\int_{0}^{\infty}\frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right\}^{p}dy=\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}}dxdy$$

$$\leq \tilde{B}\left\{\int_{0}^{\infty}(u(x))^{1-\lambda}(u'(x))^{1-p}f^{p}(x)dx\right\}^{\frac{1}{p}}\times\left\{\int_{0}^{\infty}(v(y))^{1-\lambda}(v'(y))^{1-q}g^{q}(y)dy\right\}^{\frac{1}{q}}$$

$$=\tilde{B}\left\{\int_{0}^{\infty}(u(x))^{1-\lambda}(u'(x))^{1-p}f^{p}(x)dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{0}^{\infty}(v(y))^{1-\lambda}(v'(y))^{1-q}\left[\left((v(y))^{\lambda-1}(v'(y))^{q-1}\right)\int_{0}^{\infty}\frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right]^{q(p-1)}dy\right\}^{\frac{1}{q}}$$

$$=\tilde{B}\left\{\int_{0}^{\infty}(u(x))^{1-\lambda}(u'(x))^{1-p}f^{p}(x)dx\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{0}^{\infty}\left((v(y))^{\lambda-1}(v'(y))^{q-1}\right)^{p-1}\left(\int_{0}^{\infty}\frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right)^{p}dy\right\}^{\frac{1}{q}} \tag{5.2}$$

It follows from (5.2) that the inequality (5.1) is valid after some simplifications.

On the other hand, we assume that the inequality (5.1) keeps valid**.** By applying in turn Hölder's inequality and (5.1), we have

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}}dxdy$$

$$=\int_{0}^{\infty}\left((v(y))^{\lambda-1}(v'(y))^{q-1}\right)^{\frac{1}{q}}\left\{\int_{0}^{\infty}\frac{f(x)}{(u(x)+v(y))^{\lambda}}dx\right\}\left((v(y))^{1-\lambda}(v'(y))^{1-q}\right)^{\frac{1}{q}}g(y)\,dy$$

$$\leq \left\{ \int_0^\infty \left((v(y))^{\lambda-1} (v'(y))^{q-1} \right)^{p-1} \left(\int_0^\infty \frac{f(x)}{(u(x)+v(y))^\lambda} dx \right)^p dy \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_0^\infty \left((v(y))^{1-\lambda} (v'(y))^{1-q} \right) g^q(y)\, dy \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \tilde{B}^p \int_0^\infty (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty (v(y))^{1-\lambda} (v'(y))^{1-q} g^q(y)\, dy \right\}^{\frac{1}{q}}$$

$$= \tilde{B} \left\{ \int_0^\infty (u(x))^{1-\lambda} (u'(x))^{1-p} f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty (v(y))^{1-\lambda} (v'(y))^{1-q} g^q(y)\, dy \right\}^{\frac{1}{q}} \quad (5.3)$$

If the constant factor $\tilde{B}^p$ in (5.1) is not the best possible, then it is known from (5.3) that the constant factor $\tilde{B}$ in (3.1) is also not the best possible**.** This is a contradiction. Therefore the inequality (5.1) is equivalent to (3.1)**.** Theorem is proved.

When $p = 2$, $\tilde{B}$ is reduced to$B\left(\frac{\lambda}{2},\ \frac{\lambda}{2}\right)$. According to Theorem 5.1, we have the following result.

Corollary 5.2. Let $0 < \lambda \leq 2$**,** If $\int_0^\infty \left\{ (u(x))^{1-\lambda} (u'(x))^{-1} \right\} f^2(x)\, dx < +\infty$, then

$$\int_0^\infty (v(y))^{\lambda-1} v'(y) \left(\int_0^\infty \frac{f(x)}{(u(x)+v(y))^\lambda} dx \right)^2 dy \;\leq\; \left(B(\tfrac{\lambda}{2},\ \tfrac{\lambda}{2}) \right)^2 \int_0^\infty (u(x))^{1-\lambda} (u'(x))^{-1} f^2(x)\, dx. \quad (5.4)$$

If the constant factors of the functions $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$ are not considered, then the coefficient $B\left(\frac{\lambda}{2},\ \frac{\lambda}{2}\right)$is best possible. And the inequality (5.4) is equivalent to (3.11).

In particular, when $\lambda = 1$, $\tilde{B}$ is reduced to $\frac{\pi}{\sin\frac{\pi}{p}}$. Based on Theorem 5.1, we can obtain the following result.

Corollary 5.3. Let $f(x) \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$. If $\int_0^\infty (u'(x))^{1-p} f^p(x) < +\infty$ and then

$$\int_0^\infty v'(y) \left(\int_0^\infty \frac{f(x)}{u(x)+v(y)} dx \right)^p dy \;\leq\; \left(\frac{\pi}{\sin\frac{\pi}{p}} \right)^p \int_0^\infty (u'(x))^{1-p} f^p(x)\, dx. \quad (5.5)$$

If the constant factors of the functions $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$ are not considered, then the coefficient $\tilde{B}^p$ in (5.5) is the best possible. And the inequality (5.5) is equivalent to (3.12).

Let's consider the constant factors of $u(x)$, $v(x)$, $u'(x)$ and $v'(x)$.

Theorem 5.4. *With the assumptions as Theorem 5.1, then*

$$\begin{aligned}&\int\limits_0^\infty \left\{(v(y))^{\lambda-1}(v'(y))^{q-1}\right\}^{p-1}\left(\int\limits_0^\infty \frac{f(x)}{(u(x)+v(y))^\lambda}dx\right)^p dy\\ \leq\ &(a\tilde{B})^p\int\limits_0^\infty \left\{(u(x))^{1-\lambda}(u'(x))^{1-p}\right\}f^p(x)\,dx\end{aligned} \tag{5.6}$$

where the constant factor $(a\tilde{B})^p$ *is the best possible and the constant* a *is given by (3.14),* $\tilde{u}(x)$ *and* $\tilde{v}(x)$ *as well as their derivatives are given by (3.13). And the inequality (5.6) is equivalent to (3.15).*

Its proof is similar to that of Theorem 3.5, it is omitted.

Based on the results of 3th and 4th sections, we may establish number of equivalent inequalities. Here they are omitted.

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