

# Positive Solutions for $m + 1$ - Point Discrete Boundary Value Problems

Rodica LUCA

**Abstract.** We investigate the existence of positive solutions with respect to a cone for a nonlinear system with second-order differences, subject to some boundary conditions which involve  $m + 1$  points.

**Keywords:** difference equations, boundary conditions, positive solutions, fixed point theorem.

**Mathematics Subject Classification (2000):** 39A10.

## 1 Introduction

We consider the nonlinear system with second-order differences

$$(S) \quad \begin{cases} \Delta^2 u_{n-1} + \lambda b_n f(v_n) = 0, & n = \overline{1, N-1} \\ \Delta^2 v_{n-1} + \mu c_n g(u_n) = 0, & n = \overline{1, N-1}, \quad N \geq 2, \end{cases}$$

with the  $m + 1$  - point boundary conditions

$$(BC) \quad \begin{cases} \beta u_0 - \gamma \Delta u_0 = 0, & u_N - \sum_{i=1}^{m-2} a_i u_{\xi_i} = 0, \\ \beta v_0 - \gamma \Delta v_0 = 0, & v_N - \sum_{i=1}^{m-2} a_i v_{\xi_i} = 0, \quad m \geq 3, \end{cases}$$

where  $\Delta$  is the forward difference operator with stepsize 1,  $\Delta u_n = u_{n+1} - u_n$ , and  $\overline{k, m} \stackrel{def}{=} \{k, k+1, \dots, m\}$  for  $k, m \in \mathbb{N}$ .

The above problem can be equivalently written as

$$\begin{cases} u_{n+1} - 2u_n + u_{n-1} + \lambda b_n f(v_n) = 0, & n = \overline{1, N-1} \\ v_{n+1} - 2v_n + v_{n-1} + \mu c_n g(u_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the conditions

$$\begin{cases} (\beta + \gamma)u_0 = \gamma u_1, & u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i}, \\ (\beta + \gamma)v_0 = \gamma v_1, & v_N = \sum_{i=1}^{m-2} a_i v_{\xi_i}. \end{cases}$$

In this paper we shall give sufficient conditions for  $\lambda$  and  $\mu$  such that positive solutions of (S), (BC) exist. The above problem is a generalization of the problem studied in [8], where the boundary conditions are of the form  $u_0 = v_0 = 0$ ,  $u_N = \alpha u_\eta$ ,  $v_N = \alpha v_\eta$ ,  $\eta \in \{1, 2, \dots, N-1\}$  with  $0 < \alpha < N/\eta$ , or  $u_0 - \beta \Delta u_0 = v_0 - \beta \Delta v_0 = 0$ ,  $u_N = \alpha u_\eta$ ,  $v_N = \alpha v_\eta$ . In [10] there was investigated the existence of positive solutions for the  $m$ -point boundary value problem on time scales  $u^{\Delta \nabla}(t) + a(t)f(u(t)) = 0$ ,  $t \in (0, T)$ , with boundary conditions  $\beta u(0) - \gamma u^\Delta(0) = 0$ ,  $u(T) - \sum_{i=1}^{m-2} a_i u(\xi_i) = b$ ,  $m \geq 3$ . For the study of multi-point boundary value problems for second-order or higher-order differential or difference equations we refer the reader to [1]-[7], [9], [11]-[14] and the references therein.

We shall suppose that the following conditions are satisfied

$$(A1) \quad b_n, c_n > 0, \text{ for } n = \overline{1, N-1}; \quad a_i > 0, \text{ for } i = \overline{1, m-2}, \quad a_{m-2} \geq 1; \quad \xi_i \in \mathbb{N}, \quad i = \overline{1, m-2}, \quad 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < N, \quad N > \sum_{i=1}^{m-2} a_i \xi_i; \quad \beta, \gamma \geq 0, \quad \beta + \gamma > 0, \quad d = \beta \left( N - \sum_{i=1}^{m-2} a_i \xi_i \right) + \gamma \left( 1 - \sum_{i=1}^{m-2} a_i \right) > 0.$$

(A2)  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous functions and the limits

$$f_0 = \lim_{x \rightarrow 0^+} \frac{f(x)}{x}, \quad g_0 = \lim_{x \rightarrow 0^+} \frac{g(x)}{x}, \quad f_\infty = \lim_{x \rightarrow \infty} \frac{f(x)}{x}, \quad g_\infty = \lim_{x \rightarrow \infty} \frac{g(x)}{x}$$

exist and are positive numbers.

In Section 2 we shall present several auxiliary results which investigate a discrete boundary value problem (the problem (1), (2) below). Some of these results are taken from [10]. In Section 3 we shall prove two existence theorems for the positive solutions with respect to a cone for our problem (S), (BC), which generalize Theorems 3.1 and 3.2 from [8]. These theorems are based on the Guo-Krasnoselskii fixed point theorem, presented below.

**Theorem 1.** *Let  $X$  be a Banach space and let  $C \subset X$  be a cone in  $X$ . Assume  $\Omega_1$  and  $\Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$  and let  $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow C$  be a completely continuous operator such that, either*

- i)  $\|\mathcal{A}u\| \leq \|u\|$ ,  $u \in C \cap \partial\Omega_1$ , and  $\|\mathcal{A}u\| \geq \|u\|$ ,  $u \in C \cap \partial\Omega_2$ , or
- ii)  $\|\mathcal{A}u\| \geq \|u\|$ ,  $u \in C \cap \partial\Omega_1$ , and  $\|\mathcal{A}u\| \leq \|u\|$ ,  $u \in C \cap \partial\Omega_2$ .

*Then  $\mathcal{A}$  has a fixed point in  $C \cap (\overline{\Omega_2} \setminus \Omega_1)$ .*

Finally, in Section 4 we shall give an example, that illustrates the obtained results.

## 2 Auxiliary Results

In this section we shall investigate the second-order difference system with the boundary conditions

$$\Delta^2 u_{n-1} + y_n = 0, \quad n = \overline{1, N-1}, \quad (1)$$

$$\beta u_0 - \gamma \Delta u_0 = 0, \quad u_N - \sum_{i=1}^{m-2} a_i u_{\xi_i} = 0. \quad (2)$$

**Lemma 1.** *If  $\beta + \gamma \neq 0$ ,  $0 < \xi_1 < \dots < \xi_{m-2} < N$  and  $d \neq 0$ , then the solution of (1), (2) is given by*

$$u_n = \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{n\beta + \gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j - \sum_{i=1}^{n-1} (n-i)y_i, \quad n = \overline{0, N}. \quad (3)$$

**Proof.** By system (1), for  $u_1$  fixed, we obtain

$$u_n = nu_1 - (n-1)u_0 - \sum_{i=1}^{n-1} (n-i)y_i, \quad n = \overline{2, N}. \quad (4)$$

By (2) we have  $u_0 = \frac{\gamma u_1}{\beta + \gamma}$  and  $u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i}$ . From the above relations we obtain

$$Nu_1 - (N-1)u_0 - \sum_{i=1}^{N-1} (N-i)y_i = \sum_{i=1}^{m-2} a_i \left[ \xi_i u_1 - (\xi_i - 1)u_0 - \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right]$$

or

$$\left[ N - (N-1) \frac{\gamma}{\beta + \gamma} \right] u_1 - \sum_{i=1}^{N-1} (N-i)y_i = \sum_{i=1}^{m-2} a_i \left[ \xi_i - (\xi_i - 1) \frac{\gamma}{\beta + \gamma} \right] u_1 - \sum_{i=1}^{m-2} a_i \left( \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right).$$

So we have

$$\begin{aligned} u_1 & \left\{ N - (N-1) \frac{\gamma}{\beta + \gamma} - \sum_{i=1}^{m-2} a_i \left[ \left( 1 - \frac{\gamma}{\beta + \gamma} \right) \xi_i + \frac{\gamma}{\beta + \gamma} \right] \right\} \\ & = \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \end{aligned}$$

or

$$u_1 \left( \frac{\beta N + \gamma}{\beta + \gamma} - \sum_{i=1}^{m-2} a_i \frac{\beta \xi_i + \gamma}{\beta + \gamma} \right) = \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j.$$

Therefore, because  $d \neq 0$  we obtain

$$u_1 = \frac{\beta + \gamma}{d} \left[ \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right]. \quad (5)$$

Then, by (4), (5) and  $u_0 = \gamma u_1 / (\beta + \gamma)$ , we deduce

$$\begin{aligned} u_n &= \frac{n\beta + \gamma}{\beta + \gamma} u_1 - \sum_{i=1}^{n-1} (n-i)y_i \\ &= \frac{n\beta + \gamma}{d} \left[ \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right] - \sum_{i=1}^{n-1} (n-i)y_i \\ &= \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{n\beta + \gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j - \sum_{i=1}^{n-1} (n-i)y_i, \quad n = \overline{2, N}, \\ u_1 &= \frac{\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{\beta + \gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j, \\ u_0 &= \frac{\gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{\gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j. \end{aligned}$$

We consider that  $\sum_{i=1}^0 z_i = 0$  and  $\sum_{i=1}^{-1} z_i = 0$ , and we obtain the relations (3). Q.E.D.

**Lemma 2.** *Under the assumptions of Lemma 1, the Green function for the boundary value problem (1), (2) is given by*

$$G(n, i) = \begin{cases} \frac{n\beta + \gamma}{d} (N-i) - \frac{n\beta + \gamma}{d} \sum_{k=1}^{m-2} a_k (\xi_k - i) - (n-i), & \text{if } i < \xi_1, n \geq i, \\ & (\text{for } n = 0 \text{ or } n = 1 \text{ without term } (n-i)), \\ \frac{n\beta + \gamma}{d} (N-i) - \frac{n\beta + \gamma}{d} \sum_{k=1}^{m-2} a_k (\xi_k - i), & \text{if } n \leq i < \xi_1, \\ \frac{n\beta + \gamma}{d} (N-i) - \frac{n\beta + \gamma}{d} \sum_{k=j}^{m-2} a_k (\xi_k - i) - (n-i), & \text{if } \xi_{j-1} \leq i < \xi_j, n \geq i, \\ & j = \overline{2, m-2}, \\ \frac{n\beta + \gamma}{d} (N-i) - \frac{n\beta + \gamma}{d} \sum_{k=j}^{m-2} a_k (\xi_k - i), & \text{if } \xi_{j-1} \leq i < \xi_j, n \leq i, \\ & j = \overline{2, m-2}, \\ \frac{n\beta + \gamma}{d} (N-i) - (n-i), & \text{if } \xi_{m-2} \leq i \leq n, \\ \frac{n\beta + \gamma}{d} (N-i), & \text{if } i \geq \xi_{m-2}, n \leq i. \end{cases}$$

**Proof.** By using the relations (3) we obtain

$$\begin{aligned} u_n &= \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{n\beta + \gamma}{d} \left[ a_1 \sum_{j=1}^{\xi_1-1} (\xi_1 - j)y_j + a_2 \sum_{j=1}^{\xi_2-1} (\xi_2 - j)y_j + \dots \right. \\ &\quad \left. + a_{m-2} \sum_{j=1}^{\xi_{m-2}-1} (\xi_{m-2} - j)y_j \right] - \sum_{i=1}^{n-1} (n-i)y_i \\ &= \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{n\beta + \gamma}{d} \left\{ \sum_{i=1}^{\xi_1-1} \left[ \sum_{j=1}^{m-2} a_j (\xi_j - i) \right] y_i + \sum_{i=\xi_1}^{\xi_2-1} \left[ \sum_{j=2}^{m-2} a_j (\xi_j - i) \right] y_i + \dots \right. \\ &\quad \left. + \sum_{i=\xi_{m-3}}^{\xi_{m-2}-1} a_{m-2} (\xi_{m-2} - i)y_i \right\} - \sum_{i=1}^{n-1} (n-i)y_i, \quad n = \overline{0, N}. \end{aligned}$$

Therefore we have

$$\begin{aligned} u_n &= \frac{n\beta + \gamma}{d} \left\{ \sum_{i=1}^{\xi_1-1} \left[ (N-i) - \sum_{j=1}^{m-2} a_j (\xi_j - i) \right] y_i + \sum_{i=\xi_1}^{\xi_2-1} \left[ (N-i) - \sum_{j=2}^{m-2} a_j (\xi_j - i) \right] y_i + \dots \right. \\ &\quad \left. + \sum_{i=\xi_{m-3}}^{\xi_{m-2}-1} [(N-i) - a_{m-2} (\xi_{m-2} - i)] y_i + \sum_{i=\xi_{m-2}}^{N-1} (N-i)y_i \right\} - \sum_{i=1}^{n-1} (n-i)y_i, \quad n = \overline{0, N}. \end{aligned} \tag{6}$$

By relations (6) we have  $u_n = \sum_{i=1}^{N-1} G(n, i)y_i, \quad n = \overline{0, N}$ . Q.E.D.

**Lemma 3.** If  $d > 0, \beta, \gamma \geq 0, \beta + \gamma > 0, a_i > 0$ , for all  $i = \overline{1, m-2}, 0 < \xi_1 < \dots < \xi_{m-2} < N, \sum_{i=1}^{m-2} a_i \xi_i \leq N$  and  $y_n \geq 0$ , for all  $n = \overline{1, N-1}$ , then the solution  $u_n, n = \overline{0, N}$  of problem (1), (2) satisfies  $u_n \geq 0$ , for all  $n = \overline{0, N}$ .

**Proof.** In Lemma 2.2, from [10] there was studied the problem of positive solutions for the general case of time scales. However we shall briefly present the proof in our discrete case. Because  $u_{n+1} - 2u_n + u_{n-1} = -y_n \leq 0$ , we deduce that  $u_n \geq \frac{1}{2}(u_{n+1} + u_{n-1})$ , for all  $n = \overline{1, N-1}$ , so  $(u_n)_{n=\overline{0, N}}$  is concave. To prove that  $u_n \geq 0$ , for all  $n = \overline{0, N}$ , it is sufficient to show that  $u_0, u_N \geq 0$ .

For  $u_0$  we have from (3)

$$\begin{aligned} u_0 &= \frac{\gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{\gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \geq \frac{\gamma}{d} \left[ \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{N-1} (N-j)y_j \right] \\ &= \frac{\gamma}{d} \left( 1 - \sum_{i=1}^{m-2} a_i \right) \left( \sum_{i=1}^{N-1} (N-i)y_i \right) \geq 0, \quad \text{if } \sum_{i=1}^{m-2} a_i \leq 1. \end{aligned}$$

If  $\sum_{i=1}^{m-2} a_i > 1$ , then

$$\begin{aligned} u_0 &\geq \frac{\gamma}{d} \left[ \sum_{i=1}^{N-1} (N-i)y_i - \sum_{i=1}^{m-2} a_i \sum_{j=1}^{N-1} (\xi_i - j)y_j \right] \\ &= \frac{\gamma}{d} \sum_{i=1}^{N-1} \left[ \left( N - \sum_{j=1}^{m-2} a_j \xi_j \right) + \left( \sum_{j=1}^{m-2} a_j - 1 \right) i \right] y_i \geq 0. \end{aligned}$$

For  $u_N$  we have after some computations

$$\begin{aligned} u_N &= \frac{N\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i - \frac{N\beta + \gamma}{d} \sum_{i=1}^{m-2} a_i \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j - \sum_{i=1}^{N-1} (N-i)y_i \\ &= \frac{\beta}{d} \left\{ \sum_{i=1}^{m-2} a_i \left[ \xi_i \sum_{j=1}^{N-1} (N-j)y_j - N \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right] \right\} \\ &\quad + \frac{\gamma}{d} \left\{ \sum_{i=1}^{m-2} a_i \left[ \sum_{j=1}^{N-1} (N-j)y_j - \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right] \right\} \\ &\geq \frac{\beta}{d} \left\{ \sum_{i=1}^{m-2} a_i \left\{ \sum_{j=1}^{\xi_i-1} [\xi_i(N-j) - N(\xi_i - j)]y_j + \sum_{j=\xi_i}^{N-1} \xi_i(N-j)y_j \right\} \right\} \\ &\quad + \frac{\gamma}{d} \sum_{i=1}^{m-2} a_i \left[ \sum_{j=1}^{N-1} (N-j)y_j - \sum_{j=1}^{\xi_i-1} (\xi_i - j)y_j \right] \\ &= \frac{\beta}{d} \left[ \sum_{i=1}^{m-2} a_i \left( (N - \xi_i) \sum_{j=1}^{\xi_i-1} j y_j + \xi_i \sum_{j=\xi_i}^{N-1} (N-j)y_j \right) \right] + \frac{\gamma}{d} \sum_{i=1}^{m-2} a_i (N - \xi_i) \sum_{j=1}^{N-1} y_j \geq 0. \end{aligned}$$

Q.E.D.

**Remark 1.** Under the assumptions of Lemma 3, if  $y_n > 0$ , for  $n = \overline{1, N-1}$ , then from the strict concavity of  $u_n$  we deduce that  $u_n > 0$ , for all  $n = \overline{1, N}$ . If in addition  $\gamma > 0$ , then also  $u_0 > 0$ .

**Lemma 4.** If  $d > 0$ ,  $\beta, \gamma \geq 0$ ,  $\beta + \gamma > 0$ ,  $0 < \xi_1 < \dots < \xi_{m-2} < N$ ,  $a_i > 0$ , for all  $i = \overline{1, m-2}$ ,  $a_{m-2} \geq 1$ ,  $N \geq \sum_{i=1}^{m-2} a_i \xi_i$ ,  $y_n \geq 0$ , for all  $n = \overline{1, N-1}$ , then the solution of problem (1), (2) satisfies

$$\begin{cases} u_n \leq \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i, \quad \forall n = \overline{0, N}, \\ u_{\xi_j} \geq \frac{\beta \xi_j + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i)y_i, \quad \forall j = \overline{1, m-2}. \end{cases} \quad (7)$$

**Proof.** By relations (3) we deduce that

$$u_n \leq \frac{n\beta + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i \leq \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i)y_i, \quad \forall n = \overline{0, N}.$$

Then, by using (6) we have for  $j = \overline{1, m-2}$

$$\begin{aligned} u_{\xi_j} = & \sum_{i=1}^{\xi_1-1} \left\{ \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=1}^{m-2} a_k(\xi_k - i) \right] - (\xi_j - i) \right\} y_i \\ & + \sum_{i=\xi_1}^{\xi_2-1} \left\{ \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=2}^{m-2} a_k(\xi_k - i) \right] - (\xi_j - i) \right\} y_i + \dots \\ & + \sum_{i=\xi_{j-1}}^{\xi_j-1} \left\{ \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=j}^{m-2} a_k(\xi_k - i) \right] - (\xi_j - i) \right\} y_i \\ & + \sum_{i=\xi_j}^{\xi_{j+1}-1} \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=j+1}^{m-2} a_k(\xi_k - i) \right] y_i + \dots \\ & + \sum_{i=\xi_{m-3}}^{\xi_{m-2}-1} \frac{\beta\xi_j + \gamma}{d} [(N-i) - a_{m-2}(\xi_{m-2} - i)] y_i \\ & + \sum_{i=\xi_{m-2}}^{N-1} \frac{\beta\xi_j + \gamma}{d} (N-i)y_i, \quad (\xi_0 = 1). \end{aligned}$$

We shall investigate the coefficients of  $y_i$ ,  $i = \overline{1, \xi_{m-2} - 1}$ . We obtain

$$\begin{aligned} \Lambda_1^i = & \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=1}^{m-2} a_k(\xi_k - i) \right] - (\xi_j - i) \\ = & \frac{1}{d} \left\{ (\beta\xi_j + \gamma)(N-i) - (\beta\xi_j + \gamma) \sum_{k=1}^{m-2} a_k \xi_k + (\beta\xi_j + \gamma)i \sum_{k=1}^{m-2} a_k \right. \\ & \left. - (\xi_j - i) \left[ \beta \left( N - \sum_{k=1}^{m-2} a_k \xi_k \right) + \gamma \left( 1 - \sum_{k=1}^{m-2} a_k \right) \right] \right\} \\ = & \frac{1}{d} \left[ N(\gamma + i\beta) - \xi_j(\gamma + i\beta) - (\gamma + i\beta) \sum_{k=1}^{m-2} a_k \xi_k + \xi_j(\gamma + i\beta) \right] \\ = & \frac{\gamma + i\beta}{d} \left[ N - \sum_{k=1}^{m-2} a_k \xi_k + \xi_j \left( \sum_{k=1}^{m-2} a_k - 1 \right) \right] \geq 0, \end{aligned}$$

for all  $i = \overline{1, \xi_1 - 1}$ .

For the next coefficients we have for  $p = \overline{2, j}$

$$\Lambda_p^i = \frac{\beta\xi_j + \gamma}{d} \left[ (N-i) - \sum_{k=p}^{m-2} a_k(\xi_k - i) \right] - (\xi_j - i)$$

$$\begin{aligned}
&= \frac{1}{d} \left\{ (\beta\xi_j + \gamma)(N - i) - (\beta\xi_j + \gamma) \sum_{k=p}^{m-2} a_k \xi_k + (\beta\xi_j + \gamma) i \sum_{k=p}^{m-2} a_k \right. \\
&\quad \left. - (\xi_j - i) \left[ \beta \left( N - \sum_{k=1}^{m-2} a_k \xi_k \right) + \gamma \left( 1 - \sum_{k=1}^{m-2} a_k \right) \right] \right\} \\
&\geq \frac{1}{d} \left\{ (\beta\xi_j + \gamma)(N - i) - (\beta\xi_j + \gamma) \sum_{k=p}^{m-2} a_k \xi_k + (\beta\xi_j + \gamma) i \sum_{k=p}^{m-2} a_k \right. \\
&\quad \left. - (\xi_j - i) \left[ \beta \left( N - \sum_{k=p}^{m-2} a_k \xi_k \right) + \gamma \left( 1 - \sum_{k=p}^{m-2} a_k \right) \right] \right\} \\
&= \frac{\gamma + i\beta}{d} \left[ N - \sum_{k=p}^{m-2} a_k \xi_k + \xi_j \left( \sum_{k=p}^{m-2} a_k - 1 \right) \right] \geq 0,
\end{aligned}$$

for all  $i = \overline{\xi_{p-1}, \xi_p - 1}$ .

For the last coefficients of  $y_i$  we have for  $q = \overline{j+1, m-2}$

$$\Lambda_q^i = \frac{\beta\xi_j + \gamma}{d} \left[ N - \sum_{k=q}^{m-2} a_k \xi_k + i \left( \sum_{k=q}^{m-2} a_k - 1 \right) \right] \geq 0,$$

for all  $i = \overline{\xi_{q-1}, \xi_q - 1}$ .

Therefore

$$u_{\xi_j} \geq \sum_{i=\xi_{m-2}}^{N-1} \frac{\beta\xi_j + \gamma}{d} (N - i) y_i, \quad \forall j = \overline{1, m-2}.$$

Q.E.D.

**Lemma 5.** *We assume that  $\beta, \gamma \geq 0$ ,  $\beta + \gamma > 0$ ,  $d > 0$ ,  $0 < \xi_1 < \dots < \xi_{m-2} < N$ ,  $a_i > 0$*

*for all  $i = \overline{1, m-2}$ ,  $N > \sum_{i=1}^{m-2} a_i \xi_i$  and  $y_n \geq 0$  for all  $n = \overline{1, N-1}$ . Then the unique solution*

*of problem (1), (2) verifies  $\min_{n=\xi_1, N} u_n \geq r \|u\|$ , where*

$$r = \min_{2 \leq s \leq m-2} \left\{ \frac{\xi_1}{N}, \frac{\sum_{i=1}^{m-2} a_i (N - \xi_i)}{N - \sum_{i=1}^{m-2} a_i \xi_i}, \frac{\sum_{i=1}^{m-2} a_i \xi_i}{N}, \frac{\sum_{i=1}^{s-1} a_i \xi_i + \sum_{i=s}^{m-2} a_i (N - \xi_i)}{N - \sum_{i=s}^{m-2} a_i \xi_i} \right\}$$

and  $\|u\| = \sup_{n=0, N} |u_n|$ .

For the proof of Lemma 5 see Lemma 2.4 from [10].

### 3 Existence of positive solutions of $(S)$ , $(BC)$

Using the Green function given in Lemma 2, a pair  $\left((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\right)$  is a solution of the eigenvalue problem  $(S)$ ,  $(BC)$  if and only if

$$\begin{cases} u_n = \lambda \sum_{i=1}^{N-1} G(n, i) b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right), & n = \overline{0, N} \\ v_n = \mu \sum_{i=1}^{N-1} G(n, i) c_i g(u_i), & n = \overline{0, N}. \end{cases}$$

A solution  $\left((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\right)$  of  $(S)$ ,  $(BC)$  is called a positive solution if  $u_i > 0$ ,  $v_i > 0$ , for all  $i = \overline{1, N-1}$ .

We consider the Banach space  $X = \mathbb{R}^{N+1}$  with supremum norm  $\|\cdot\|$  and define the cone  $C \subset X$  by

$$C = \left\{ u = (u_0, \dots, u_N) \in X, u_i \geq 0, \forall i = \overline{0, N} \text{ and } \min_{n=\xi_1, N} u_n \geq r \|u\| \right\},$$

where  $r$  is defined in Lemma 5.

We define the positive numbers  $L_1$  and  $L_2$  by

$$L_1 = \max \left\{ \left( r^2 \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) b_j f_\infty \right)^{-1}, \left( r^2 \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j g_\infty \right)^{-1} \right\},$$

$$L_2 = \min \left\{ \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) b_j f_0 \right)^{-1}, \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j g_0 \right)^{-1} \right\}.$$

**Theorem 2.** *Assume the assumptions (A1), (A2) hold and  $L_1 < L_2$ . Then for each  $\lambda$  and  $\mu$  that satisfy  $\lambda, \mu \in (L_1, L_2)$ , there exists a solution  $\left((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\right)$  for  $(S)$ ,  $(BC)$ , with  $u_n > 0$ ,  $v_n > 0$ , for all  $n = \overline{1, N-1}$ .*

**Proof.** In the proof we shall follow the same steps as those from Theorem 3.1 in [8]. Let  $\lambda$  and  $\mu$  such that  $\lambda, \mu \in (L_1, L_2)$  and we select a positive number  $\varepsilon$  such that  $\varepsilon < f_\infty$ ,  $\varepsilon < g_\infty$ ,

$$\max \left\{ \left( r^2 \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) b_j (f_\infty - \varepsilon) \right)^{-1}, \left( r^2 \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j (g_\infty - \varepsilon) \right)^{-1} \right\} \leq \min(\lambda, \mu)$$

and

$$\max(\lambda, \mu) \leq \min \left\{ \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) b_j (f_0 + \varepsilon) \right)^{-1}, \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j (g_0 + \varepsilon) \right)^{-1} \right\}.$$

We now define the operator  $\mathcal{A} : C \rightarrow X$ , by

$$\mathcal{A}(u) = \left( \lambda \sum_{i=1}^{N-1} G(n, i) b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right) \right)_{n=\overline{0, N}}, \quad u = (u_n)_{n=\overline{0, N}} \in C.$$

We have  $\mathcal{A}(C) \subset C$ . Indeed, if  $u = (u_n)_{n=\overline{0, N}} \in C$ , then  $v = (v_n)_{n=\overline{0, N}}$ , with  $v_n = \mathcal{A}(u)_n$  is the solution of (1) – (2) with  $y_n = \lambda b_n f \left( \mu \sum_{i=1}^{N-1} G(n, i) c_i g(u_i) \right) \geq 0$ , for all  $n = \overline{1, N-1}$ . So by Lemma 3 we have  $v_n \geq 0$ , for all  $n = \overline{0, N}$  and by Lemma 5 we deduce that  $\min_{n=\overline{\xi_1, N}} v_n \geq r \|v\|$ . Therefore  $v = \mathcal{A}(u) \in C$ . Then it is easy to verify that  $\mathcal{A}$  is completely continuous (compact and continuous). By definitions of  $f_0$  and  $g_0$ , there exists  $K_1 > 0$  such that

$$f(x) \leq (f_0 + \varepsilon)x \quad \text{and} \quad g(x) \leq (g_0 + \varepsilon)x, \quad 0 < x \leq K_1.$$

By (A2) we have  $f(0) = g(0) = 0$  and the above inequalities are also valid for  $x = 0$ .

Let  $u \in C$  with  $\|u\| = K_1$ . Because  $v_i = \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j)$ ,  $i = \overline{0, N}$  verify the problem (1), (2) with  $y_n = \mu c_n g(u_n)$ ,  $n = \overline{1, N-1}$ , then by (7) and the above property of  $g$ , we deduce for all  $i = \overline{0, N}$

$$\begin{aligned} v_i &= \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \leq \mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j g(u_j) \\ &\leq \mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j (g_0 + \varepsilon) u_j \\ &\leq \mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j (g_0 + \varepsilon) \|u\| \leq \|u\| = K_1. \end{aligned}$$

By using once again Lemma 4 (relations (7)) and the properties of function  $f$  we have

$$\begin{aligned} \mathcal{A}(u)_n &= \lambda \sum_{i=1}^{N-1} G(n, i) b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right) \\ &\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right) \\ &\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i (f_0 + \varepsilon) \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \end{aligned}$$

$$\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i)b_i(f_0 + \varepsilon)K_1 \leq K_1 = \|u\|, \quad \forall n = \overline{0, N}.$$

Then  $\|\mathcal{A}(u)\| \leq \|u\|$ , for all  $u \in C$  with  $\|u\| = K_1$ . We denote by  $\Omega_1 = \{u \in C, \|u\| < K_1\}$ . So  $\|\mathcal{A}(u)\| \leq \|u\|$ , for all  $u \in C \cap \partial\Omega_1$ .

By definition of  $f_\infty$  and  $g_\infty$ , there exists  $\bar{K}_2 > 0$  such that

$$f(x) \geq (f_\infty - \varepsilon)x \quad \text{and} \quad g(x) \geq (g_\infty - \varepsilon)x, \quad x \geq \bar{K}_2.$$

We consider now  $K_2 = \max\left\{2K_1, \frac{\bar{K}_2}{r}\right\}$ . For  $u \in C$  with  $\|u\| = K_2$ , we obtain by using Lemma 5, that

$$u_n \geq \min_{n=\xi_1, \bar{N}} u_n \geq r\|u\| = rK_2 \geq \bar{K}_2, \quad \text{for all } n = \overline{\xi_1, \bar{N}}.$$

Then, by using (7), Lemma 5 and the above considerations, we obtain for  $i \geq \xi_1$

$$\begin{aligned} v_i &= \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \geq r\|v\| \geq rv_{\xi_{m-2}} = r\mu \sum_{j=1}^{N-1} G(\xi_{m-2}, j)c_j g(u_j) \\ &\geq r\mu \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j g(u_j) \\ &\geq r\mu \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j (g_\infty - \varepsilon)u_j \\ &\geq r^2\mu \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j (g_\infty - \varepsilon)\|u\| \geq \|u\| = K_2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}(u)_{\xi_{m-2}} &\geq \lambda \sum_{i=\xi_{m-2}}^{N-1} \frac{\beta \xi_{m-2} + \gamma}{d} (N-i)b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \right) \\ &\geq \lambda \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i)b_i (f_\infty - \varepsilon) \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \\ &\geq \lambda \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i)b_i (f_\infty - \varepsilon) K_2 \\ &\geq \lambda r^2 \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i)b_i (f_\infty - \varepsilon) K_2 \geq K_2 = \|u\|. \end{aligned}$$

Therefore  $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)_{\xi_{m-2}} \geq \|u\|$ , for all  $u \in C$  with  $\|u\| = K_2$ . We denote by  $\Omega_2 = \{u \in C, \|u\| < K_2\}$ . Then  $\|\mathcal{A}(u)\| \geq \|u\|$ , for all  $u \in C \cap \partial\Omega_2$ .

Now we can apply Theorem 1, i) and we deduce that  $\mathcal{A}$  has a fixed point  $u \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$ . Because  $\|u\| > 0$  and  $u$  is concave ( $\Delta^2 u_{n-1} \leq 0$  for all  $n = \overline{1, N-1}$ ) we deduce that  $u_n > 0$

for all  $n = \overline{1, N-1}$ . This element  $u$  together with  $v_n = \mu \sum_{i=1}^{N-1} G(n, i) c_i g(u_i)$ ,  $n = \overline{0, N}$  represent a solution of  $(S)$ ,  $(BC)$ . In addition  $\|v\| > 0$  and by concavity of  $v$  we have  $v_n > 0$  for all  $n = \overline{1, N-1}$ . Indeed if  $v_n = 0$  for all  $n = \overline{0, N}$ , then by  $(S)$ ,  $(BC)$  and Lemma 1 we obtain  $u_n = 0$  for all  $n = \overline{0, N}$ , which is false. So  $(u, v)$  is a positive solution for our problem  $(S) - (BC)$ . Q.E.D.

**Remark 2.** The condition  $L_1 < L_2$  from Theorem 2 is equivalent to

$$\frac{d}{r^2(\beta\xi_{m-2} + \gamma)} \left( \min \left\{ \sum_{j=\xi_{m-2}}^{N-1} (N-j)b_j f_\infty, \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j g_\infty \right\} \right)^{-1} < \frac{d}{\beta N + \gamma} \left( \max \left\{ \sum_{j=1}^{N-1} (N-j)b_j f_0, \sum_{j=1}^{N-1} (N-j)c_j g_0 \right\} \right)^{-1}$$

or

$$\frac{\max \left\{ \sum_{j=1}^{N-1} (N-j)b_j f_0, \sum_{j=1}^{N-1} (N-j)c_j g_0 \right\}}{\min \left\{ \sum_{j=\xi_{m-2}}^{N-1} (N-j)b_j f_\infty, \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j g_\infty \right\}} < \frac{r^2(\beta\xi_{m-2} + \gamma)}{\beta N + \gamma}.$$

**Remark 3.** From the above proof we see that the assumption (A2) in Theorem 2 can be replace by the weaker one

(A2)'  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous functions and the supremum/infimum limits

$$\tilde{f}_0 = \limsup_{x \rightarrow 0^+} \frac{f(x)}{x}, \quad \tilde{g}_0 = \limsup_{x \rightarrow 0^+} \frac{g(x)}{x}, \quad \tilde{f}_\infty = \liminf_{x \rightarrow \infty} \frac{f(x)}{x}, \quad \tilde{g}_\infty = \liminf_{x \rightarrow \infty} \frac{g(x)}{x}$$

exist and are positive numbers.

In what follows we shall present another existence result, that generalizes Theorem 3.2 from [8]. Let us consider the positive numbers

$$L_3 = \max \left\{ \left( r \frac{\beta\xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)b_j f_0 \right)^{-1}, \left( r \frac{\beta\xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j g_0 \right)^{-1} \right\},$$

$$L_4 = \min \left\{ \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)b_j f_\infty \right)^{-1}, \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)c_j g_\infty \right)^{-1} \right\}.$$

**Theorem 3.** Assume the assumptions (A1)-(A2) hold and  $L_3 < L_4$ . Then for each  $\lambda$  and  $\mu$  that satisfy  $\lambda, \mu \in (L_3, L_4)$ , there exists a solution  $((u_n)_{n=\overline{0, N}}, (v_n)_{n=\overline{0, N}})$  for  $(S)$ ,  $(BC)$ , with  $u_n > 0$ ,  $v_n > 0$ , for all  $n = \overline{1, N-1}$ .

**Proof.** Let  $\lambda$  and  $\mu$  with  $\lambda, \mu \in (L_3, L_4)$ . We select a positive number  $\varepsilon$  such that  $\varepsilon < f_0$ ,  $\varepsilon < g_0$  and

$$\max \left\{ \left( r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)b_j(f_0 - \varepsilon) \right)^{-1}, \right. \\ \left. \left( r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j)c_j(g_0 - \varepsilon) \right)^{-1} \right\} \leq \min(\lambda, \mu)$$

and

$$\max(\lambda, \mu) \leq \min \left\{ \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)b_j(f_\infty + \varepsilon) \right)^{-1}, \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)c_j(g_\infty + \varepsilon) \right)^{-1} \right\}.$$

We also consider the operator  $\mathcal{A}$  defined in the proof of Theorem 2. From the definitions of  $f_0$  and  $g_0$ , we deduce that there exists  $\bar{K}_3 > 0$  such that

$$f(x) \geq (f_0 - \varepsilon)x \quad \text{and} \quad g(x) \geq (g_0 - \varepsilon)x, \quad 0 < x \leq \bar{K}_3.$$

By (A2) we have  $f(0) = g(0) = 0$  and the above inequalities are also valid for  $x = 0$ .

In addition, because  $g$  is a continuous function with  $g_0 > 0$ , then  $g(0) = 0$  and there exists  $K_3 \in (0, \bar{K}_3)$  such that

$$g(x) \leq \frac{\bar{K}_3}{\mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)c_j}, \quad 0 < x \leq K_3.$$

For  $u \in C$  with  $\|u\| = K_3$ , by (7) and the above inequality, we deduce that for all  $i \in \overline{0, N}$

$$v_i = \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \leq \mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)c_j g(u_j) \\ \leq \mu \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j)c_j \frac{\bar{K}_3}{\mu \frac{\beta N + \gamma}{d} \sum_{k=1}^{N-1} (N-k)c_k} = \bar{K}_3.$$

By using (7), Lemma 5, the choice of  $\lambda, \mu$ , the facts that  $v, u \in C$  and  $v_i, u_j \leq \bar{K}_3$  for  $i, j \in \overline{0, N}$ , we then obtain

$$\mathcal{A}(u)_{\xi_{m-2}} \geq \lambda \sum_{i=\xi_{m-2}}^{N-1} \frac{\beta \xi_{m-2} + \gamma}{d} (N-i)b_i f \left( \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \right) \\ \geq \lambda \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i)b_i (f_0 - \varepsilon) \left( \mu \sum_{j=1}^{N-1} G(i, j)c_j g(u_j) \right)$$

$$\begin{aligned}
&\geq \lambda \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) b_i (f_0 - \varepsilon) r \|v\| \\
&\geq \lambda \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) b_i (f_0 - \varepsilon) r v_{\xi_{m-2}} \\
&\geq \left( \lambda r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) b_i (f_0 - \varepsilon) \right) \cdot \left( \mu \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j g(u_j) \right) \\
&\geq \left( \lambda r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) b_i (f_0 - \varepsilon) \right) \cdot \left( \mu \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j (g_0 - \varepsilon) u_j \right) \\
&\geq \left( \lambda r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{i=\xi_{m-2}}^{N-1} (N-i) b_i (f_0 - \varepsilon) \right) \cdot \left( \mu r \frac{\beta \xi_{m-2} + \gamma}{d} \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j (g_0 - \varepsilon) \right) \|u\| \\
&\geq \|u\|.
\end{aligned}$$

Hence  $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)_{\xi_{m-2}} \geq \|u\|$ , for  $u \in C$  with  $\|u\| = K_3$ . We denote by  $\Omega_3 = \{u \in C, \|u\| < K_3\}$ , and then we have  $\|\mathcal{A}(u)\| \geq \|u\|$  for all  $u \in C \cap \partial\Omega_3$ .

We now consider the functions  $f^*, g^* : [0, \infty) \rightarrow [0, \infty)$ , defined by  $f^*(x) = \sup_{0 \leq y \leq x} f(y)$ ,  $g^*(x) = \sup_{0 \leq y \leq x} g(y)$ . By (A2) we obtain for  $f^*$  and  $g^*$  the relations

$$\lim_{x \rightarrow \infty} \frac{f^*(x)}{x} = f_\infty, \quad \lim_{x \rightarrow \infty} \frac{g^*(x)}{x} = g_\infty.$$

We also have  $f(x) \leq f^*(x)$ ,  $g(x) \leq g^*(x)$ , for all  $x \geq 0$ . Then there exists  $\bar{K}_4 > 0$  such that

$$f^*(x) \leq (f_\infty + \varepsilon)x, \quad g^*(x) \leq (g_\infty + \varepsilon)x, \quad \text{for all } x \geq \bar{K}_4.$$

Let  $K_4 > \max\{2K_3, \bar{K}_4\}$ . Then for  $u$  with  $\|u\| = K_4$  we obtain

$$\begin{aligned}
\mathcal{A}(u)_n &\leq \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i \lambda f \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right) \\
&\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^* \left( \mu \sum_{j=1}^{N-1} G(i, j) c_j g(u_j) \right) \\
&\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^* \left( \frac{\beta N + \gamma}{d} \mu \sum_{j=1}^{N-1} (N-j) c_j g(u_j) \right) \\
&\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^* \left( \frac{\beta N + \gamma}{d} \mu \sum_{j=1}^{N-1} (N-j) c_j g^*(u_j) \right) \\
&\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^* \left( \frac{\beta N + \gamma}{d} \mu \sum_{j=1}^{N-1} (N-j) c_j g^*(K_4) \right)
\end{aligned}$$

$$\begin{aligned}
 &\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^* \left( \frac{\beta N + \gamma}{d} \mu \sum_{j=1}^{N-1} (N-j) c_j (g_\infty + \varepsilon) K_4 \right) \\
 &\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i f^*(K_4) \\
 &\leq \lambda \frac{\beta N + \gamma}{d} \sum_{i=1}^{N-1} (N-i) b_i (f_\infty + \varepsilon) K_4 \leq K_4 = \|u\|, \quad \forall n = \overline{0, N}.
 \end{aligned}$$

So  $\|\mathcal{A}(u)\| \leq \|u\|$ , for all  $u \in C$  with  $\|u\| = K_4$ . If we denote by  $\Omega_4 = \{u \in C, \|u\| < K_4\}$ , then we obtain  $\|\mathcal{A}(u)\| \leq \|u\|$ , for all  $u \in C \cap \partial\Omega_4$ .

By Theorem 1 ii) we deduce that  $\mathcal{A}$  has a fixed point  $u \in C \cap (\overline{\Omega_4} \setminus \Omega_3)$ , which together with  $v_n = \mu \sum_{i=1}^{N-1} G(n, i) c_i g(u_i)$ ,  $n = \overline{0, N}$ , give us a positive solution for (S), (BC). Q.E.D.

**Remark 4.** The condition  $L_3 < L_4$  is equivalent to

$$\begin{aligned}
 &\frac{d}{r(\beta \xi_{m-2} + \gamma)} \left( \min \left\{ \sum_{j=\xi_{m-2}}^{N-1} (N-j) b_j f_0, \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j g_0 \right\} \right)^{-1} \\
 &< \frac{d}{\beta N + \gamma} \left( \max \left\{ \sum_{j=1}^{N-1} (N-j) b_j f_\infty, \sum_{j=1}^{N-1} (N-j) c_j g_\infty \right\} \right)^{-1}
 \end{aligned}$$

or

$$\frac{\max \left\{ \sum_{j=1}^{N-1} (N-j) b_j f_\infty, \sum_{j=1}^{N-1} (N-j) c_j g_\infty \right\}}{\min \left\{ \sum_{j=\xi_{m-2}}^{N-1} (N-j) b_j f_0, \sum_{j=\xi_{m-2}}^{N-1} (N-j) c_j g_0 \right\}} < \frac{r(\beta \xi_{m-2} + \gamma)}{\beta N + \gamma}.$$

**Remark 5.** The assumption (A2) in Theorem 3 can be replace by

(A2)<sup>n</sup>  $f, g : [0, \infty) \rightarrow [0, \infty)$  are continuous functions and the infimum/supremum limits

$$\tilde{f}_0 = \liminf_{x \rightarrow 0^+} \frac{f(x)}{x}, \quad \tilde{g}_0 = \liminf_{x \rightarrow 0^+} \frac{g(x)}{x}, \quad \tilde{f}_\infty = \limsup_{x \rightarrow \infty} \frac{f(x)}{x}, \quad \tilde{g}_\infty = \limsup_{x \rightarrow \infty} \frac{g(x)}{x}$$

exist and are positive numbers.

The last two relations imply  $\limsup_{x \rightarrow \infty} \frac{f^*(x)}{x} = \tilde{f}_\infty$  and  $\limsup_{x \rightarrow \infty} \frac{g^*(x)}{x} = \tilde{g}_\infty$ .

## 4 An example

As in Example 4.1 in [8], let us consider the functions

$$\begin{cases} f(x) = p_2 |\sin x| + p_1 x e^{-1/x}, & x \in [0, \infty), \\ g(x) = q_2 |\sin x| + q_1 x e^{-1/x}, & x \in [0, \infty), \end{cases}$$

with  $p_1, p_2, q_1, q_2 > 0$ .

We have  $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = p_2$ ,  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = p_1$ ,  $\lim_{x \rightarrow 0^+} \frac{g(x)}{x} = q_2$ ,  $\lim_{x \rightarrow \infty} \frac{g(x)}{x} = q_1$ .

Let  $b_n, c_n$  defined by  $b_n = \frac{b_0}{N-n}$ ,  $c_n = \frac{c_0}{N-n}$ , for all  $n = \overline{1, N-1}$ , with  $b_0 > 0$ ,  $c_0 > 0$ , and  $\beta = 2$ ,  $\gamma = 1/3$ ,  $m = 2$ ,  $\xi_1 = \frac{N}{3}$ ,  $\xi_2 = \frac{N}{2}$ , ( $N = 6M$ ,  $M \in \mathbb{N}$ ),  $a_1 = a_2 = 1$ .

We consider the system with second-order differences

$$(S_0) \quad \begin{cases} \Delta^2 u_{n-1} + \lambda \frac{b_0}{N-n} \left[ p_2 |\sin v_n| + p_1 v_n e^{-1/v_n} \right] = 0, & n = \overline{1, N-1} \\ \Delta^2 v_{n-1} + \mu \frac{c_0}{N-n} \left[ q_2 |\sin u_n| + q_1 u_n e^{-1/u_n} \right] = 0, & n = \overline{1, N-1}, \quad N \geq 6, \end{cases}$$

with the boundary conditions

$$(BC_0) \quad \begin{cases} u_0 = \frac{u_1}{7}, & u_N = u_{N/3} + u_{N/2}, \\ v_0 = \frac{v_1}{7}, & v_N = v_{N/3} + v_{N/2}. \end{cases}$$

In this case the number  $d$  is

$$d = 2(N - (a_1 \xi_1 + a_2 \xi_2)) + \frac{1}{3}(1 - a_1 - a_2) = \frac{N-1}{3} > 0.$$

Also the condition  $N > a_1 \xi_1 + a_2 \xi_2$  is verified, because it is equivalent to  $N > \frac{5N}{6}$ , and

$$r = \min \left\{ \frac{\xi_1}{N}, \frac{a_1(N - \xi_1) + a_2(N - \xi_2)}{N - (a_1 \xi_1 + a_2 \xi_2)}, \frac{a_1 \xi_1 + a_2 \xi_2}{N}, \frac{a_1 \xi_1 + a_2(N - \xi_2)}{N - a_2 \xi_2} \right\} = \frac{1}{3}.$$

The numbers  $L_1$  and  $L_2$  from Section 3 are of the form

$$\begin{aligned} L_1 &= \max \left\{ \left( r^2 \frac{\beta \xi_2 + \gamma}{d} \sum_{j=\xi_2}^{N-1} (N-j) b_j p_1 \right)^{-1}, \left( r^2 \frac{\beta \xi_2 + \gamma}{d} \sum_{j=\xi_2}^{N-1} (N-j) c_j q_1 \right)^{-1} \right\} \\ &= \max \left\{ \left( \frac{3N+1}{9(N-1)} \sum_{j=N/2}^{N-1} b_0 p_1 \right)^{-1}, \left( \frac{3N+1}{9(N-1)} \sum_{j=N/2}^{N-1} c_0 q_1 \right)^{-1} \right\} \\ &= \frac{18(N-1)}{N(3N+1)} \max\{(b_0 p_1)^{-1}, (c_0 q_1)^{-1}\} = \frac{18(N-1)}{N(3N+1)} (\min\{b_0 p_1, c_0 q_1\})^{-1}, \\ L_2 &= \min \left\{ \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) b_j p_2 \right)^{-1}, \left( \frac{\beta N + \gamma}{d} \sum_{j=1}^{N-1} (N-j) c_j q_2 \right)^{-1} \right\} \\ &= \min \left\{ \left( \frac{6N+1}{N-1} \sum_{j=1}^{N-1} b_0 p_2 \right)^{-1}, \left( \frac{6N+1}{N-1} \sum_{j=1}^{N-1} c_0 q_2 \right)^{-1} \right\} \\ &= \frac{1}{6N+1} \min\{(b_0 p_2)^{-1}, (c_0 q_2)^{-1}\} = \frac{1}{6N+1} (\max\{b_0 p_2, c_0 q_2\})^{-1} \end{aligned}$$

and the condition  $L_1 < L_2$  becomes

$$\frac{\max\{b_0 p_2, c_0 q_2\}}{\min\{b_0 p_1, c_0 q_1\}} < \frac{N(3N + 1)}{18(N - 1)(6N + 1)}.$$

Therefore, if the above condition is verified, then by Theorem 2 we deduce that for all positive numbers  $\lambda, \mu \in (L_1, L_2)$  the problem  $(S_0)$ ,  $(BC_0)$  has positive solutions.

## References

- [1] D.R. Anderson, *Solutions to second-order three-point problems on time scales*, J. Differ. Equations Appl., 8 (2002), 673-688.
- [2] D.R. Anderson, *Twin  $n$ -point boundary value problems*, Appl. Math. Lett., 17 (2004), 1053-1059.
- [3] R. Avery, *Three positive solutions of a discrete second order conjugate problem*, PanAmer. Math. J., 8 (1998), 79-96.
- [4] W. Cheung, J. Ren, *Positive solutions for discrete three-point boundary value problems*, Austr. J. Math. Anal. Appl., 1 (2004), 1-7.
- [5] Y. Guo, W. Shan, W. Ge, *Positive solutions for second order  $m$ -point boundary value problems*, J. Comput. Appl. Math., 151 (2003), 415-424.
- [6] J. Henderson, S.K. Ntouyas, *Positive solutions for systems of nonlinear boundary value problems*, Nonlinear Stud., 15 (2008), 51-60.
- [7] J. Henderson, S.K. Ntouyas, *Positive solutions for systems of three-point nonlinear boundary value problems*, Austr. J. Math. Anal. Appl., 5 (2008), (1), 1-9.
- [8] J. Henderson, S.K. Ntouyas, I.K. Purnaras, *Positive solutions for systems of three-point nonlinear discrete boundary value problems*, Neural, Parallel and Scientific Computations, 16 (2008), 209-224.
- [9] V. Il'in, E. Moiseev, *Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Equations, 23 (1987), 803-810.
- [10] W.T. Li, H.R. Sun, *Positive solutions for second-order  $m$ -point boundary value problems on times scales*, Acta Mathematica Sinica, English Series, 22, No.6 (2006), 1797-1804.
- [11] R. Ma, *Positive solutions for second order three-point boundary value problems*, Appl. Math. Lett., 14 (2001), 1-5.
- [12] R. Ma, Y. Raffoul, *Positive solutions of three-point nonlinear discrete second order boundary value problem*, J. Difference Equ. Appl., 10 (2004), 129-138.

- [13] S.K. Ntouyas, *Nonlocal initial and boundary value problems: a survey*, *Handbook of differential equations: Ordinary differential equations*, Vol.II, 461-557, Elsevier, Amsterdam, 2005.
- [14] H.R. Sun, W.T. Li, *Existence of positive solutions for nonlinear three-point boundary value problems on time scales*, *J. Math. Anal. Appl.*, 299 (2004), 508-524.