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On Weak s-m-Continuity for Multifunctions

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Abstract. In this paper we introduce and investigate weakly s-m-continuous multifunctions as a generalization of both weakly m-continuous multifunctions and s-m-continuous multifunctions. The class of weakly s-m-continuous multifunctions contains weakly s-precontinuous multifunctions due to Ekici and Park [6] as a special case.

Keywords: m-structure, weakly s-m-continuous, weakly s-precontinuous, multifunction.

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1 Introduction

Semi-open sets, preopen sets, α-open sets, b-open sets and β-open sets play an important role in the research of generalizations of continuity in topological spaces. By using these sets, many authors introduced and investigated various types of non-continuous functions and multifunctions. In 1978, Kohli [9] defined a function $f : X \to Y$ to be s-continuous if for each point $x \in X$ and each open set V of Y containing $f(x)$ and having connected complement, there exists an open set U of X containing x such that $f(U) \subset V$. In 1989, Lipski [13] extended this notion to the setting of multifunctions. By replacing open sets of X with semi-open (resp. preopen, β-open) sets, Ewert and Lipski [8] (resp. Popa and Noiri [21], [22]) defined and investigated upper/lower s-quasi-continuous (resp. upper/lower s-precontinuous, upper/lower s-β-continuous) multifunctions. In [18], [23] and [25], the present authors introduced the notions of m-structures, m-spaces, m-continuous functions and multifunctions. By using these notions, a unified theory for S-continuity of multifunctions was formulated in [24]. On the other hand, in 1961 Levine [11] introduced the concept of weakly continuous functions which was extended to the setting of multifunctions in [20] and [26]. Recently, the present authors [19] introduced and studied the notions of upper/lower weakly m-continuous multifunctions. As a generalization of weakly continuous multifunctions and s-precontinuous multifunctions, Ekici and Park [6] introduced and studied weakly s-precontinuous multifunctions.

In this paper, we introduce the notions of upper/lower weakly s-m-continuous multifunctions as a generalization of upper/lower s-m-continuous multifunctions and upper/lower weakly m-continuous multifunctions. We obtain some characterizations and several properties of such multifunctions.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) = A$).

Definition 2.1 Let (X, τ) be a topological space. A subset A of X is said to be
(1) *α-open* [17] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [12] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [15] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) *b-open* [3] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all α-open (resp. semi-open, preopen, β-open, b-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Definition 2.2 Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [16] (resp. *semi-closed* [5], *preclosed* [15], *β-closed* [1], *b-closed* [3]) if the complement of A is α-open (resp. semi-open, preopen, β-open, b-open).

Definition 2.3 Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, β-closed, b-closed) sets of X containing A is called the *α-closure* [16] (resp. *semi-closure* [5], *preclosure* [7], *β-closure* [2], *b-closure* [3]) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$).

Definition 2.4 Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, β-open, b-open) sets of X contained in A is called the *α-interior* [16] (resp. *semi-interior* [5], *preinterior* [7], *β-interior* [2], *b-interior* [3]) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, ${}_{\beta}\mathrm{Int}(A)$, $\mathrm{bInt}(A)$).

A point $x \in X$ is called a *θ-cluster* point of a subset A of X [27] if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed [27]. The complement of a θ-closed set is said to be θ-open. The union of all θ-open sets contained in A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It is shown in [27] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(A)$ is closed in X for each subset A of X.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ (or simply $F : X \to Y$) presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5 A multifunction $F : X \to Y$ is said to be
(1) *upper s-continuous* [13] (resp. *upper s-quasi-continuous* [8], *upper s-precontinuous* [21], *upper s-β-continuous* [22]) at a point $x \in X$ if for each open set V containing $F(x)$ and

having connected complement, there exists an open (resp. semi-open, preopen, β-open) set $U \subset X$ containing x such that $F(U) \subset V$,

(2) *lower s-continuous* [13] (resp. *lower s-quasi-continuous* [8], *lower s-precontinuous* [21], *lower s-β-continuous* [22]) at a point $x \in X$ if for each open set V of Y meeting $F(x)$ and having connected complement , there exists an open (resp. semi-open, preopen, β-open) set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper (lower) s-continuous* (resp. *upper (lower) s-quasi-continuous*, *upper (lower) s-precontinuous*, *upper (lower) s-β-continuous*) in X if it has this property at every point of X.

Definition 2.6 A multifunction $F : X \to Y$ is said to be

(1) *upper weakly s-precontinuous* [6] if for each point $x \in X$ and each open set V containing $F(x)$ and having connected complement, there exists a preopen set $U \subset X$ containing x such that $F(U) \subset \mathrm{Cl}(V)$,

(2) *lower weakly s-precontinuous* [6] if for each point $x \in X$ and each open set V of Y meeting $F(x)$ and having connected complement , there exists a preopen set $U \subset X$ containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

3 Weakly s-m-continuous multifunctions

Definition 3.1 A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of a m_X-open set is said to be *m_X-closed*.

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are all m-structures on X.

Definition 3.2 Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [14] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2 Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$, $\beta(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, ${}_{\beta}\mathrm{Int}(A)$).

Lemma 3.1 (Maki et al. [14]). *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,

(2) If $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 3.2 (Popa and Noiri [23]) *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 3.3 An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [14] if the union of any family of subsets belong to m_X belongs to m_X.

Remark 3.3 Let (X, τ) be a topological space. Then τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ all satisfy property $\mathcal{B}$.

Lemma 3.3 (Popa and Noiri [18]). *Let X be a nonempty set and m_X an m-structure on X satisfying $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,

(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ and $\mathrm{mCl}(A)$ is m_X-closed.

Definition 3.4 Let (X, m_X) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper s-m-continuous* [24] (resp. *upper weakly s-m-continuous*) at $x \in X$ if for each $V \in \sigma$ containing $F(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $F(U) \subset V$ (resp. $F(U) \subset \mathrm{Cl}(V)$),

(2) *lower s-m-continuous* [24] (resp. *lower weakly s-m-continuous*) at $x \in X$ if for each $V \in \sigma$ meeting $F(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ (resp. $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$) for each $u \in U$,

(3) *upper/lower s-m-continuous* (resp. *upper/lower weakly s-m-continuous*) if it has this property at each point x of X.

Remark 3.4 Every upper/lower s-m-continuous multifunction is upper/lower weakly s-m-continuous. Example 6 of [6] shows that this implication is not reversible.

Remark 3.5 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by the definition of upper/lower weakly s-m-continuous multifunctions, we obtain the definition of upper/lower weakly s-precontinuous multifunctions due to Ekici and Park [6] stated in Definition 2.6.

Theorem 3.1 (Popa and Noiri [24]). *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper s-m-continuous;

(2) $F^+(V) = \mathrm{mInt}(F^+(V))$ for each open set V of Y having connected complement;

(3) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every connected closed set K of Y;

(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having the connected closure;

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is connected.

Theorem 3.2 (Popa and Noiri [24]). *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following are equivalent:*

(1) F is lower s-m-continuous;

(2) $F^-(V) = \mathrm{mInt}(F^-(V))$ *for each open set* V *of* Y *having connected complement;*
(3) $F^+(K) = \mathrm{mCl}(F^+(K))$ *for every connected closed set* K *of* Y*;*
(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the connected closure;*
(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is connected.*

Theorem 3.3 *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) F *is upper weakly* s-m*-continuous;*
(2) $F^+(G) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(G)))$ *for each open set* G *of* Y *having connected complement;*
(3) $\mathrm{mCl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every connected closed set* K *of* Y*;*
(4) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B))$ *for every subset* B *of* Y *having the connected closure;*
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset* B *of* Y *such that* $Y - \mathrm{Int}(B)$ *is connected;*
(6) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^-(\mathrm{Cl}(G))$ *for every open set* G *of* Y *having the connected closure;*
(7) $\mathrm{mCl}(F^-(G))) \subset F^-(\mathrm{Cl}(G))$ *for every open set* G *of* Y *having the connected closure;*
(8) $\mathrm{mCl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every connected regular closed set* K *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let G be any open set of Y having connected complement and $x \in F^+(G)$. Then $F(x) \subset G$. There exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Cl}(G)$. Therefore, we have $x \in U \subset F^+(\mathrm{Cl}(G))$. Since $U \in m_X$, we have $x \in \mathrm{mInt}(F^+(\mathrm{Cl}(G)))$ and hence $F^+(G) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(G)))$.

(2) $\Rightarrow$ (3): Let K be a connected closed set of Y. Then $Y - K$ is an open set in Y having connected complement. By (2) and Lemma 3.1, we have

$$X - F^-(K) = F^+(Y - K) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(Y - K))) = \mathrm{mInt}(F^+(Y - \mathrm{Int}(K))) = \mathrm{mInt}(X - F^-(\mathrm{Int}(K))) = X - \mathrm{mCl}(F^-(\mathrm{Int}(K))).$$

Therefore, we obtain $\mathrm{mCl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y having the connected closure. Then $\mathrm{Cl}(B)$ is closed connected in Y and by (3) we have $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y such that $Y - \mathrm{Int}(B)$ is connected. Then by (4) and Lemma 3.1, we have

$$X - \mathrm{mInt}(F^+(\mathrm{Cl}(\mathrm{Int}(B)))) = \mathrm{mCl}(X - F^+(\mathrm{Cl}(\mathrm{Int}(B)))) = \mathrm{mCl}(F^-(Y - \mathrm{Cl}(\mathrm{Int}(B)))) = \mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(Y - B)))) \subset F^-(\mathrm{Cl}(Y - B)) = X - F^+(\mathrm{Int}(B)).$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and G be any open set of Y having connected complement such that $F(x) \subset G$. Then $x \in F^+(G) = F^+(\mathrm{Int}(G)) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(G)))$. Then, there exists $U \in m_X$ containing x such that $x \in U \subset F^+(\mathrm{Cl}(G))$. Therefore, we obtain $F(U) \subset \mathrm{Cl}(G)$ and hence F is upper weakly s-m-continuous.

(4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7): They are obvious.

(7) $\Rightarrow$ (8): Let K be any connected regular closed set of Y. Then $K = \mathrm{Cl}(\mathrm{Int}(K))$ is connected and by (7) we have $\mathrm{mCl}(F^-(\mathrm{Int}(K))) \subset F^-(\mathrm{Cl}(\mathrm{Int}(K))) = F^-(K)$.

(8) $\Rightarrow$ (3): Let K be any connected closed set of Y. Since K is connected, $\mathrm{Int}(K)$ is connected and hence $\mathrm{Cl}(\mathrm{Int}(K))$ is connected. Let $H = \mathrm{Cl}(\mathrm{Int}(K))$, then H is a regular closed connected set of Y and $\mathrm{Int}(H) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K))) = \mathrm{Int}(K)$. By (8), we have $\mathrm{mCl}(F^-(\mathrm{Int}(K))) = \mathrm{mCl}(F^-(\mathrm{Int}(H))) \subset F^-(H) \subset F^-(K)$.

Theorem 3.4 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly s-m-continuous;

(2) $F^-(G) \subset \mathrm{mInt}(F^-(\mathrm{Cl}(G)))$ for each open set G of Y having connected complement;

(3) $\mathrm{mCl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ for every connected closed set K of Y;

(4) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y having the connected closure;

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(\mathrm{Cl}(\mathrm{Int}(B))))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is connected;

(6) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^+(\mathrm{Cl}(G))$ for every open set G of Y having the connected closure;

(7) $\mathrm{mCl}(F^+(G))) \subset F^+(\mathrm{Cl}(G))$ for every open set G of Y having the connected closure;

(8) $\mathrm{mCl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ for every connected regular closed set K of Y.

Proof. The proof is similar to that of Theorem 3.3.

Remark 3.6 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then, by Theorems 3.3 and 3.4, we obtain the results established in Theorems 4 and 7 of [6].

Theorem 3.5 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper weakly s-m-continuous;

(2) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^-(\mathrm{Cl}(G))$ for every β-open set G of Y having the connected closure;

(3) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^-(\mathrm{Cl}(G))$ for every semi-open set G of Y having the connected closure.

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 3.3(4).

(2) $\Rightarrow$ (3): The proof is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.

(3) $\Rightarrow$ (1): Since $\sigma \subset \mathrm{SO}(Y)$, the proof follows from Theorem 3.3(7).

Theorem 3.6 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly s-m-continuous;

(2) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^+(\mathrm{Cl}(G))$ for every β-open set G of Y having the connected closure;

(3) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^+(\mathrm{Cl}(G))$ for every semi-open set G of Y having the connected closure.

Proof. The proof is similar to that of Theorem 3.5.

Theorem 3.7 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper weakly s-m-continuous;

(2) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset B of Y having the connected θ-closure;*

(3) $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset B of Y having the connected θ-closure.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y having the connected θ-closure. Then $\mathrm{Cl}_\theta(B)$ is connected closed and by Theorem 3.3 we obtain $\mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$.

(2) $\Rightarrow$ (3): The proof is obvious since $\mathrm{Cl}(B) \subset \mathrm{Cl}_\theta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (1): Let K be a regular closed connected set of Y. Then we have

$$\mathrm{Cl}_\theta(\mathrm{Int}(K)) = \mathrm{Cl}(\mathrm{Int}(K)) = K \text{ and } \mathrm{mCl}(F^-(\mathrm{Int}(K))) = \mathrm{mCl}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K))))) \subset F^-(\mathrm{Cl}_\theta(\mathrm{Int}(K))) = F^-(\mathrm{Cl}(\mathrm{Int}(K))) = F^-(K).$$

Therefore, we have $\mathrm{mCl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ and by Theorem 3.3(8) f is upper weakly s-m-continuous.

Theorem 3.8 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly s-m-continuous;

(2) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B))$ *for every subset B of Y having the connected θ-closure;*

(3) $\mathrm{mCl}(F^+(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^+(\mathrm{Cl}_\theta(B))$ *for every subset B of Y having the connected θ-closure.*

Proof. The proof is similar to that of Theorem 3.7.

Definition 3.5 A subset A of a topological space (X, τ) is said to be
(1) *α-paracompact* [28] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,
(2) *α-regular* [10] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, a multifunction $\mathrm{Cl}F : (X, m_X) \to (Y, \sigma)$ is defined in [4] as follows: $(\mathrm{Cl}F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$, ${}_\beta\mathrm{Cl}(F)$.

Lemma 3.4 (Noiri and Popa [19]). *If $F : (X, m_X) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then*

(1) $F^+(V) = G^+(V)$ for each open set V of Y,

(2) $F^-(K) = G^-(K)$ for each closed set K of Y,

where G denotes $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* ${}_\beta\mathrm{Cl}F$.

Lemma 3.5 (Noiri and Popa [19]). *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

(1) $F^-(V) = G^-(V)$ *for each open set* V *of* Y*,*
(2) $F^+(K) = G^+(K)$ *for each closed set* K *of* Y*,*
where G *denotes* $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* ${}_\beta\mathrm{Cl}F$.

Theorem 3.9 *Let* $F : (X, m_X) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α*-regular and* α*-paracompact for each* $x \in X$*. Then* F *is upper weakly* s*-*m*-continuous if and only if* $G : (X, m_X) \to (Y, \sigma)$ *is upper weakly* s*-*m*-continuous, where* G *denotes* $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* ${}_\beta\mathrm{Cl}F$.

Proof. We set $G = \mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$, ${}_\beta\mathrm{Cl}F$.

Necessity. Suppose that F is upper weakly s-m-continuous. Let V be any open set of Y having connected complement. By Theorem 3.3 and Lemmas 3.4 and 3.5, we have $G^+(V) = F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(V))) = \mathrm{mInt}(G^+(\mathrm{Cl}(V)))$. Therefore, by Theorem 3.3 G is upper weakly s-m-continuous.

Sufficiency. Suppose that G is upper weakly s-m-continuous. Let V be any open set of Y having connected complement. By Theorem 3.3 and Lemmas 3.4 and 3.5, we have $F^+(V) = G^+(V) =\subset \mathrm{mInt}(G^+(\mathrm{Cl}(V))) = \mathrm{mInt}(F^+(\mathrm{Cl}(V)))$. Therefore, by Theorem 3.3 F is upper weakly s-m-continuous.

Theorem 3.10 *Let* $F : (X, m_X) \to (Y, \sigma)$ *be a multifunction such that* $F(x)$ *is* α*-regular and* α*-paracompact for each* $x \in X$*. Then* F *is lower weakly* s*-*m*-continuous if and only if* $G : (X, m_X) \to (Y, \sigma)$ *is lower weakly* s*-*m*-continuous, where* G *denotes* $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* ${}_\beta\mathrm{Cl}F$.

Proof. The proof is similar to that of Theorem 3.9.

Remark 3.7 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorems 3.9 and 3.10, we obtain the results established in Theorems 18 and 19 of [6].

4 Weak s-m-continuity and s-m-continuity

Theorem 4.1 *Let* $F : (X, m_X) \to (Y, \sigma)$ *be a multifunction such that F(x) is open in Y for each* $x \in X$*, then F is lower* s*-*m*-continuous if and only if F is lower weakly* s*-*m*-continuous.*

Proof. Suppose that F is lower weakly s-m-continuous. Let $x \in X$ and V be any open set of Y having connected complement. There exists $U \in m_X$ containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower s-m-continuous. The converse is obvious by Remark 3.4.

Remark 4.1 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorems 4.1 , we obtain the result established in Theorem 9 of [6].

Theorem 4.2 *If* $F : (X, m_X) \to (Y, \sigma)$ *is lower weakly* s*-*m*-continuous and there exists an open basis* $\{V_\alpha : \alpha \in \Lambda\}$ *of* σ *such that* V_α *has connected complement and* $F^-(\mathrm{Cl}(V_\alpha)) \subset F^-(V_\alpha)$ *for each* $\alpha \in \Lambda$*, then* F *is lower* s*-*m*-continuous.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open basis of σ such that V_α has connected complement and $F^-(\mathrm{Cl}(V_\alpha)) \subset F^-(V_\alpha)$ for each $\alpha \in \Lambda$. For any open set V having connected complement, there exists a subset Λ_0 of Λ such that $V = \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore, by Theorem 3.4 we obtain

$$F^-(V) = F^-(\cup_{\alpha\in\Lambda_0} V_\alpha) = \cup_{\alpha\in\Lambda_0} F^-(V_\alpha) \subset \cup_{\alpha\in\Lambda_0} \mathrm{mInt}(F^-(\mathrm{Cl}(V_\alpha))) \subset$$
$$\cup_{\alpha\in\Lambda_0} \mathrm{mInt}(F^-(V_\alpha)) \subset \mathrm{mInt}(\cup_{\alpha\in\Lambda_0} F^-(V_\alpha)) \subset \mathrm{mInt}(F^-(\cup_{\alpha\in\Lambda_0} V_\alpha)) = \mathrm{mInt}(F^-(V))$$

Therefore, we obtain $F^-(V) \subset \mathrm{mInt}(F^-(V))$. By Lemma 3.1, $F^-(V) = \mathrm{mInt}(F^-(V))$ and by Theorem 3.2 F is lower s-m-continuous.

Remark 4.2 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorems 4.2 , we obtain the result established in Theorem 9 of [6].

Theorem 4.3 *If $F : (X, m_X) \to (Y, \sigma)$ is upper weakly s-m-continuous and satisfies $F^+(\mathrm{Cl}(V)) \subset F^+(V)$ for every open set V of Y having connected complement, then F is upper s-m-continuous.*

Proof. Let V be any open set of Y having connected complement. Since F is upper weakly s-m-continuous, by Theorem 3.3 we have $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Cl}(V))) \subset \mathrm{mInt}(F^+(V))$. By Lemma 3.1, $F^+(V) = \mathrm{mInt}(F^+(V))$ and by Theorem 3.1 F is upper s-m-continuous.

Remark 4.3 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorems 4.3 , we obtain the result established in Theorem 10 of [6].

Definition 4.1 A topological space (X, τ) is said to be *s-normal* [6] if for each disjoint closed sets K and F of X, there exist open sets U and V having connected complements such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

Theorem 4.4 *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that F(x) is closed in Y for each $x \in X$ and Y is s-normal. Then F is upper weakly s-m-continuous if and only if F is upper s-m-continuous.*

Proof. Suppose that F is upper weakly s-m-continuous. Let $x \in X$ and G be an open set of Y containing $F(x)$ and having connected complement. Since $F(x)$ is closed in Y, by the s-normality of Y there exist open sets V and W having connected complements such that $F(x) \subset V, Y - G \subset W$ and $V \cap W = \emptyset$. We have$F(x) \subset V \subset \mathrm{Cl}(V) \subset \mathrm{Cl}(Y - W) = Y - W$. Since F is upper weakly s-m-continuous, there exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper s-m-continuous.

Remark 4.4 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorems 4.4, we obtain the result established in Theorem 12 of [6].

Definition 4.2 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *weakly* m-continuous* [19] if $X - F^-(\mathrm{Fr}(V)) \in m_X$ for each open set V of Y.

Theorem 4.5 *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$. If a multifunction $F : (X, m_X^2) \to (Y, \sigma)$ is upper weakly s-m-continuous and $F : (X, m_X^1) \to (Y, \sigma)$ is weakly* m-continuous, then $F : (X, m_X^2) \to (Y, \sigma)$ is upper s-m-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $F(x)$ having connected complement. There exists $G \in m_X^2$ containing x such that $F(G) \subset \mathrm{Cl}(V)$. Put $U = G \cap (X - F^-(\mathrm{Fr}(V)))$. Then $U \in m_X^2$. Moreover, we have $F(x) \cap \mathrm{Fr}(V) \subset V \cap \mathrm{Cl}(V) \cap (Y - V) = \emptyset$ and hence $x \in X - F^-(\mathrm{Fr}(V))$. Therefore, $x \in U$ and $F(U) \subset V$ since $F(U) \subset F(G) \subset \mathrm{Cl}(V)$ and $F(u) \cap \mathrm{Fr}(V) = \emptyset$ for each $u \in U$. Therefore, $F : (X, m_X^2) \to (Y, \sigma)$ is upper s-m-continuous.

Theorem 4.6 *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$. If a multifunction $F : (X, m_X^1) \to (Y, \sigma)$ is upper weakly s-m-continuous and $F : (X, m_X^2) \to (Y, \sigma)$ is weakly* m-continuous, then $F : (X, m_X^2) \to (Y, \sigma)$ is upper s-m-continuous.*

Proof. The proof is similar to that of Theorem 4.5.

5 Some properties of weakly s-m-continuity

Definition 5.1 A topological space (X, τ) is said to be *strongly s-normal* [6] if for every disjoint closed sets K and F of X, there exist open sets U and V having connected complements such that $K \subset U, F \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.

Definition 5.2 An m-space (X, m_X) is said to be m-T_2 [25] if for every distinct points x and y of X, there exist m_X-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 5.3 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *injective* if $x \neq y$ implies that $F(x) \cap F(y) = \emptyset$.

Theorem 5.1 *If $F : (X, m_X) \to (Y, \sigma)$ is an injective upper weakly s-m-continuous multifunction into a strongly s-normal space (Y, σ) and F(x) is closed for each $x \in X$, then X is m-T_2.*

Proof. For any distinct points x, y of X, we have $F(x) \cap F(y) = \emptyset$ since F is injective. Since $F(x)$ is closed for each $x \in X$ and Y is strongly s-normal, there exist open sets V_1, V_2 having connected complements such that $F(x) \subset V_1, F(y) \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F is upper weakly s-m-continuous, there exist $U_1, U_2 \in m_X$ containing x, y, respectively, such that $F(U_1) \subset \mathrm{Cl}(V_1)$ and $F(U_2) \subset \mathrm{Cl}(V_2)$. Therefore, we have $U_1 \cap U_2 = \emptyset$ and hence X is m-T_2.

Remark 5.1 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \mathrm{PO}(X)$. Then by Theorem 5.1, we obtain the result established in Theorem 31 of [6].

Definition 5.4 A subset A of an m-space (X, m_X) is said to be *m-dense* in X [19] if $\mathrm{mCl}(A) = X$.

Definition 5.5 A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be *upper weakly m-continuous* [19] if for each point $x \in X$ and each open set V containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset \mathrm{Cl}(V)$.

Theorem 5.2 *Let X be a nonempty set with two m-structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ whenever $U \in m_X^1$ and $V \in m_X^2$ and (Y, σ) be a strongly s-normal space.*
If the following four conditions are satisfied,
(1) a multifunction $F : (X, m_X^1) \to (Y, \sigma)$ is upper weakly s-m-continuous,
(2) a multifunction $G : (X, m_X^2) \to (Y, \sigma)$ is upper weakly m-continuous,
(3) F(x) and G(x) are closed sets of (Y, σ) for each $x \in X$,
(4) $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,
then $A = m_X^2\mathrm{Cl}(A)$. Furthermore, if $F(x) \cap G(x) \neq \emptyset$ for each point x in an m-dense set D of (X, m_X^2), then $F(x) \cap G(x) \neq \emptyset$ for each point x in X.

Proof. Suppose that $x \in X - A$. Then we have $F(x) \cap G(x) = \emptyset$. Since (Y, σ) is strongly s-normal, there exist open sets V and W with connected complements such that $F(x) \subset V, G(x) \subset W$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(W) = \emptyset$. Since F is upper weakly s-m-continuous, there exists $U_1 \in m_X^1$ containing x such that $F(U_1) \subset \mathrm{Cl}(V)$. Since G is upper weakly m-continuous, there exists $U_2 \in m_X^2$ containing x such that $G(U_2) \subset \mathrm{Cl}(W)$. Now set $U = U_1 \cap U_2$, then we have $x \in U \in m_X^2$ and $U \cap A = \emptyset$. By Lemma 3.2, we have $x \in X - m_X^2\mathrm{Cl}(A)$ and hence $A = m_X^2\mathrm{Cl}(A)$. On the other hand, $F(x) \cap G(x) \neq \emptyset$ on D and hence $D \subset A$. Since D is m-dense on (X, m_X^2), we have $X = m_X^2\mathrm{Cl}(D) \subset m_X^2\mathrm{Cl}(A) = A$. Therefore, we obtain that $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$.

Corollary 5.1 (Ekici and Park [6]). *Let (Y, σ) be a strongly s-normal space and $F, G : (X, \tau) \to (Y, \sigma)$ upper weakly s-precontinuous and upper weakly continuous, respectively, and point closed multifunctions. Then the set $\{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is preclosed in X.*

Definition 5.6 A topological space (X, τ) is said to be *s-connected* if X cannot be written as the union of two nonempty disjoint open sets having connected complements.

Definition 5.7 An m-space (X, m_X) is said to be *m-connected* [18] if X cannot be written as the union of two nonempty disjoint m_X-open sets.

Theorem 5.3 *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. If $F : (X, m_X) \to (Y, \sigma)$ is an upper weakly s-m-continuous (or lower weakly s-m-continuous) surjective multifunction such that F(x) is connected for each $x \in X$ and (X, m_X) is m-connnected, then (Y, σ) is s-connected.*

Proof. Suppose that (Y, σ) is not s-connected. Then, there exist nonempty open sets U, V having connected complements such that $Y = U \cup V$ and $U \cap V = \emptyset$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:
(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,
(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$,
(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.
Next, we shall show that $F^+(U)$ and $F^+(V)$ are m_X-open.
(i) Let F be upper weakly s-m-continuous. By Theorem 3.3, we obtain $F^+(V) \subset$

$\text{mInt}(F^+(\text{Cl}(V))) = \text{mInt}(F^+(V))$ since V is clopen. Therefore, by Lemmas 3.1 and 3.3, we obtain $F^+(V) \in m_X$. Similarly, we obtain $F^+(U) \in m_X$. This shows that (X, m_X) is not m-connected.

(ii) Let F be lower weakly s-m-continuous. Since V is a clopen set with connected complement, by Theorem 3.4 we obtain $\text{mCl}(F^+(V)) \subset F^+(\text{Cl}(V)) = F^+(V)$. Therefore, by Lemma 3.1, we obtain $F^+(V) = \text{mCl}(F^+(V))$ and by Lemma 3.3 $F^+(V)$ is m-closed. Thus $F^+(U) \in m_X$. Similarly, we obtain $F^+(V) \in m_X$. This shows that (X, m_X) is not m-connected.
(i) and (ii) complete the proof.

Definition 5.8 Let (X, m_X) be an m-space and A be a subset of X. The *m-frontier* of A, denoted by $\text{mFr}(A)$, is defined by $\text{mFr}(A) = \text{mCl}(A) \cap \text{mCl}(X - A) = \text{mCl}(A)\text{-mInt}(A)$.

Theorem 5.4 *Let (X, m_X) be an m-space and (Y, σ) a topological space. The set of all points x of X at which a multifunction $F : (X, m_X) \to (Y, \sigma)$ is not upper weakly s-m-continuous (resp. lower weakly s-m-continuous) is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of the closure of open sets containing (resp. meeting) $F(x)$ and having connected complement.*

Proof. Let x be a point of X at which F is not upper weakly s-m-continuous. Then, there exists an open set V of Y containing $F(x)$ and having connected complement such that $U \cap (X - F^+(\text{Cl}(V))) \neq \emptyset$ for every $U \in m_X$ containing x. By Lemma 3.2, we have $x \in \text{mCl}(X - F^+(V))$ and hence $x \in \text{mFr}(F^+(V))$ since $x \in F^+(V) \subset \text{mCl}(F^+(V))$.

Conversely, if F is upper weakly s-m-continuous at x, then for every open set V of Y containing $F(x)$ and having connected complement, there exists $U \in m_X$ containing x such that $F(U) \subset \text{Cl}(V)$; hence $U \subset F^+(\text{Cl}(V))$. Therefore, we obtain $x \in U \subset \text{mInt}(F^+(\text{Cl}(V)))$. This contradicts that $x \in \text{mFr}(F^+(\text{Cl}(V)))$. In case F is lower weakly s-m-continuous the proof is similar.

Remark 5.2 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and $m_X = \text{PO}(X)$. Then by Theorem 5.4, we obtain the result established in Theorem 21 of [6].

6 New forms of upper/lower weak s-m-continuity

In Remark 3.5, we point out that if $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, $m_X = \text{PO}(X)$ and $F : (X, m_X) \to (Y, \sigma)$ is upper/lower weakly s-m-continuous, then F is said to be upper/lower weakly s-precontinuous [6]. Analogously, we can define new multifunctions as follows:

Definition 6.1 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be *upper/lower weakly s-continuous* (resp. *upper/lower weakly s-α-continuous*, *upper/lower weakly quasi s-continuous*, *upper/lower weakly s-b-continuous*, *upper/lower weakly s-β-continuous*) if $m_X = \tau$ (resp. $\alpha(X)$, $\text{SO}(X)$, $\text{BO}(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ is upper/lower weakly s-m-continuous.

All results concerning upper/lower weakly s-m-continuous multifunctions obtained in Sections 3-5 are applied to the multifunctions in Definition 6.1. For example, in case $m_X = \tau$ we obtain the characterizations of upper/lower weakly s-continuous multifunctions from Theorems 3.3 and 3.4:

Theorem 6.1 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper weakly s-continuous;

(2) $F^+(G) \subset \mathrm{Int}(F^+(\mathrm{Cl}(G)))$ *for each open set G of Y having connected complement;*

(3) $\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every connected closed set K of Y;*

(4) $\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^-(\mathrm{Cl}(B))$ *for every subset B of Y having the connected closure;*

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{Int}(F^+(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset B of Y such that* $Y - \mathrm{Int}(B)$ *is connected;*

(6) $\mathrm{Cl}(F^-(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^-(\mathrm{Cl}(G))$ *for every open set G of Y having the connected closure;*

(7) $\mathrm{Cl}(F^-(G))) \subset F^-(\mathrm{Cl}(G))$ *for every open set G of Y having the connected closure;*

(8) $\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every connected regular closed set K of Y.*

Theorem 6.2 *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower weakly s-continuous;

(2) $F^-(G) \subset \mathrm{Int}(F^-(\mathrm{Cl}(G)))$ *for each open set G of Y having connected complement;*

(3) $\mathrm{Cl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ *for every connected closed set K of Y;*

(4) $\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y having the connected closure;*

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{Int}(F^-(\mathrm{Cl}(\mathrm{Int}(B))))$ *for every subset B of Y such that* $Y - \mathrm{Int}(B)$ *is connected;*

(6) $\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(G)))) \subset F^+(\mathrm{Cl}(G))$ *for every open set G of Y having the connected closure;*

(7) $\mathrm{Cl}(F^+(G))) \subset F^+(\mathrm{Cl}(G))$ *for every open set G of Y having the connected closure;*

(8) $\mathrm{Cl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ *for every connected regular closed set K of Y.*

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