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Apriori Error Bounds in Linear Elasticity

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Abstract. The present paper presents some a priori error bounds for linear elasticity. It presents the equations of the linear elasticity, and the main features of error analysis. The present paper introduces Cea's lemma in a perturbed case for a weak approximate form of the first fundamental problem of linear elasticity.

Keywords: Finite element method, linear elasticity, error analysis, apriori bounds.

1 Introduction

In this paper we will present the equations of linear elasticity and a discussion over some a priori error bounds in linear elasticity. The main result behind the theory presented here is Cea's lemma, the novelty of this paper consisting in its implementation in a perturbed case. The flowchart of this paper is the following: in section 2 the basic equations of linear elastostatics are presented following the ideas from [1] (M.E.Gurtin). Section 3 presents the weak formulation of the equations presented in section 2 and the main result of this paper: Cea's lemma in a perturbed case. The a priori bound obtained is given by the formula (3.24).

2 Basic equations

Let $D = (0,1) \times (0,1)$ be a bounded domain in the bi-dimensional Euclidian space. Suppose that the domain D is occupied by an isotropic and homogenous medium. As in [4] (P.P.Teodorescu), the basic equations of equilibrium of the linear elasticity are:

- the equilibrium equations:

$$t_{\beta\alpha,\beta} + \rho_0 f_\alpha = 0 \quad \text{on} \quad D \tag{2.1}$$

- the constitutive equations:

$$t_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \lambda\varepsilon_{rr}\delta_{\alpha\beta} \quad \text{on} \quad D \tag{2.2}$$

- the strain-displacement equations:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \quad \text{on} \quad D \tag{2.3}$$

where, u_α are the components of the displacement vector, $t_{\alpha\beta}$ are the components of the stress tensor, $\varepsilon_{\alpha\beta}$ are the components of the strain tensor, f_α are the components of the specific body force, λ, μ are Lame constants, characteristic of the material. We'll attach the following boundary conditions:

$$u_\alpha = \tilde{u}_\alpha \quad \text{on} \quad \Gamma \tag{2.4}$$

where $\tilde{u}_\alpha$ are continuous given functions on $\Gamma = \partial D$. Thus the first fundamental boundary value problem of linear elasticity consists in finding u_α which satisfy (2.1)-(2.3) and the boundary conditions (2.4).

After some elementary computations, we can state the problem:

$$\mu u_{\alpha,\beta\beta} + (\lambda + \mu)\, u_{\beta,\beta\alpha} + f_\alpha = 0 \quad \text{on} \quad D \tag{2.5}$$

$$u_\alpha = \tilde{u}_\alpha \quad \text{on} \quad \Gamma, \ \alpha = 1, 2, \ \beta = 1, 2. \tag{2.6}$$

Considering the formula:

$$\Delta \mathbf{u} = \nabla div \ \mathbf{u} - rot \ rot \ \mathbf{u} \tag{2.7}$$

the equations (2.5) become:

$$(\lambda + 2\mu)\, \Delta div \ \mathbf{u} = -div \ \mathbf{f} \tag{2.8}$$

If we apply the operators *div* and *rot* then to the equation (2.8) we obtain:

$$(\lambda + 2\mu)\, \Delta div \ \mathbf{u} = -div \ \mathbf{f} \tag{2.9}$$

and

$$\mu \Delta rot \ \mathbf{u} = -rot \ \mathbf{f} \tag{2.10}$$

Let's make the following notation:

$$\mathbf{v} = \mu \ \mathbf{u} + \frac{1}{2}(\lambda + \mu)\, \mathbf{p} \ div \ \mathbf{u}, \quad \text{where} \ \ \mathbf{p}(\mathbf{x}) = \mathbf{x} - \mathbf{0}. \tag{2.11}$$

Applying the Laplace operator to (2.11), we can write:

$$\Delta \mathbf{v} = \mu \Delta \mathbf{u} + (\lambda + \mu) \left\{ \nabla div \ \mathbf{u} + \frac{1}{2}\mathbf{p} \ (\Delta div \ \mathbf{u}) \right\} \tag{2.12}$$

In this way ([1]), from (2.8), (2.9) and (2.12) we obtain:

$$\Delta \mathbf{v} = -\mathbf{f} - \frac{\lambda + \mu}{2(\lambda + 2\mu)} \mathbf{p} \ div \ \mathbf{f} \tag{2.13}$$

Thus, by changing accordingly the variables, the first fundamental problem of linear elasticity becomes:

$$-\Delta \mathbf{v} = \mathbf{f} \quad \text{on} \quad D \tag{2.14}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma \tag{2.15}$$

The boundary condition is also obtained using a corresponding variable transformation.

3 Weak formulation. The finite element method

In order to obtain the weak form of our problem we will define the following solutions space S:

$$S = \{ \mathrm{v} \mid \mathrm{v} \in H^1(D),\ \mathrm{v} = \mathbf{0} \ \text{ on } \ \Gamma \} \tag{3.1}$$

Let's suppose that the solution is continuous on $\bar{D}$ and thus we'll have: $S \equiv H_0^1(D)$. In this case the existence and uniqueness of the weak solution is an immediate consequence of the Lax- Milgram theorem applied on the space S.

Let's consider a weight function $\mathrm{w} \in H_0^1(D)$. Multiplying the equation (2.14) by w and integrating the resulting equation on D, we can state the weak form of the first fundamental problem of linear elasticity:let's determine $\mathbf{u} \in S$ so that:

$$\int_D \mathrm{w}\ \Delta\mathbf{u}\ dD = \int_D \mathrm{w}\ \mathbf{f}\ dD, \quad \forall \mathrm{w} \in H_0^1(D) \tag{3.2}$$

Let's define the following functionals on $H_0^1(D) \times H_0^1(D)$, and respectively on $H_0^1(D)$:

$$B\,(\mathbf{w}, \mathbf{u}) = \int_D (\mathbf{w}_{,x}\ \mathbf{u}_{,x} + \mathbf{w}_{,y}\ \mathbf{u}_{,y}\,)\ dxdy \tag{3.3}$$

$$l(\mathrm{w}) = \int_D \mathrm{w}\ \mathbf{f}\ dxdy \tag{3.4}$$

In this way, the weak form of the problem (2.14), (2.15) can be stated: let's determine $\mathbf{u} \in S$ such that:

$$B(\mathrm{w}, \mathrm{u}) = l(\mathrm{w}), \quad \forall \mathrm{w} \in H_0^1(D) \tag{3.5}$$

In order to implement the finite element method, let's consider the finite dimensional subspace $S^h \subset S$ and the approximation $\mathbf{u}^h \in S^h$. In order to generate the mesh on D, let's consider the nodes: $x_i = ih, i = 1, \ldots, N$ si $y_i = ih, i = 1, \ldots, N$ where $h = \frac{1}{N}$. Let's denote: T_{i1} the triangle with the verices (x_i, y_j), (x_i, y_{j+1}), (x_{i+1}, y_j), T_{i2} the triangle with the verices (x_i, y_j), (x_i, y_{j+1}), (x_{i-1}, y_{j+1}), T_{i3} the triangle with the verices (x_i, y_j), (x_{i-1}, y_j), (x_{i+1}, y_{j+1}), T_{i4} the triangle with the verices (x_i, y_j), (x_i, y_{j-1}), (x_{i-1}, y_j), T_{i5} the triangle with the verices (x_i, y_j), (x_i, y_{j-1}), (x_{i+1}, y_{j-1}), T_{i6} the triangle with the verices (x_i, y_j), (x_{i+1}, y_j), (x_{i+1}, y_{j-1}). In this notations we can consider the following base of shape functions in S^h:

$$\Phi_{ij}(x, y) = \begin{cases} 1 - \dfrac{x - x_i}{h} - \dfrac{y - y_j}{h},\ (x, y) \in T_{1i} \\ 1 - \dfrac{y - y_j}{h},\ (x, y) \in T_{i2} \\ 1 - \dfrac{x_i - x}{h},\ (x, y) \in T_{i3} \\ 1 - \dfrac{x_i - x}{h} - \dfrac{y_j - y}{h},\ (x, y) \in T_{i4} \\ 1 - \dfrac{y_j - y}{h},\ (x, y) \in T_{i5} \\ 1 - \dfrac{x - x_i}{h},\ (x, y) \in (x, y) \in T_{i6} \\ 0,\ \text{in all other cases} \end{cases} \tag{3.6}$$

The approximative weak form of the problem can be stated: given $\mathbf{f}, \bar{\mathbf{u}}$ and h , let's determine $\mathbf{u}^h$ such that:

$$B\left(\mathbf{w}^h, \mathbf{u}^h\right) = l\left(\mathbf{w}^h\right), \quad \forall \mathbf{w}^h \in H_0^1(D) \tag{3.7}$$

Next we recall a famous result in the elliptical equations theory: Céa's lemma ([3]):

Céa's lemma. *If* $\mathbf{u} \in S$ *and* $\mathbf{u}^h \in S^h$ *then we have:*

$$||\mathbf{u} - \mathbf{u}^h||_{H^1(D)} \le C \min_{\mathbf{v}^h \in S^h} ||\mathbf{u} - \mathbf{v}^h||_{H^1(D)}, \ C > 0, \text{ const.} \tag{3.8}$$

The demonstration of the lemma is immediate and we'll discuss it only in a self-adjoint case in the following. Another classic result in the error analysis of the finite element theory is:

$$\min_{\mathbf{v}^h \in S^h} ||\mathbf{u} - \mathbf{v}^h|| \le h^{\min(k, r-1)} ||\mathbf{u}||_{H^1(D)} \tag{3.9}$$

where k is the order of the polynomials from the finite element basis space.

The equation (3.5) is self-adjoint because the elasticity tensor $\mathbf{C}$ is positive definite. In addition, we have:

$$B(\mathbf{v}, \mathbf{v}) \ge k \ ||\mathbf{v}||^2_{H^1(D)}, \quad \forall \mathbf{v} \in H_0^1(D) \tag{3.10}$$

Let's define on $H_0^1(D) \times H_0^1(D)$ the following bilinear symmetric form:

$$[\mathbf{u}, \mathbf{v}]_B = B(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in H_0^1(D) \tag{3.11}$$

One can notice immediately that $[\ ,\ ]_B$ is a scalar product on $H_0^1(D)$. Let's consider the energetic associate norm $||\mathbf{u}||_B$:

$$||\mathbf{u}||_B = (B(\mathbf{u}, \mathbf{u}))^{\frac{1}{2}}. \tag{3.12}$$

More details about the energetic norm associated with a positive definite bilinear form can be found in [2] (S.G.Mihlin).We have $S^h \subset H_0^1(D)$, if we take $\mathbf{w} = \mathbf{w}^h \in S^h$ in (3.5) we obtain:

$$B(\mathbf{w}^h, \mathbf{u}) = l(\mathbf{w}^h), \quad \forall \mathbf{w}^h \in S^h \tag{3.13}$$

On the other hand, the approximative weak form of the problem (3.5) is:

$$B(\mathbf{w}^h, \mathbf{u}^h) = l(\mathbf{w}^h), \quad \forall \mathbf{w}^h \in S^h \tag{3.14}$$

From (3.13) and (3.14) we obtain the notorious Galerkin ortogonality relation:

$$B(\mathbf{u} - \mathbf{u}^h, \mathbf{w}^h) = 0, \quad \forall \mathbf{w}^h \in S^h \tag{3.15}$$

which can be written:

$$\left[\mathbf{u} - \mathbf{u}^h, \mathbf{w}^h\right]_B = 0, \quad \forall \mathbf{w}^h \in S^h \tag{3.16}$$

Let's prove Céa lemma in a self-adjoint case: if $u \in S$ and $u^h \in S^h$ then we have the formula:

$$||\mathbf{u} - \mathbf{u}^h||_B = \min_{\mathbf{v}^h \in S^h} ||\mathbf{u} - \mathbf{v}^h||_B. \tag{3.17}$$

From (3.15) we have:

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}^h||_B^2 &= \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h\right)_B = \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u}\right)_B - \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u}^h\right)_B \\ &= \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u}\right)_B = \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u}\right)_B - \left(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h\right)_B \\ &= \left(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h\right)_B, \quad \forall \mathbf{v}^h \in S^h \end{aligned} \tag{3.18}$$

On the other hand, from (3.18) and the Cauchy-Schwarz inequality we have:

$$||\mathbf{u}-\mathbf{u}^h||_B^2 = \left(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h\right)_B \le ||\mathbf{u}-\mathbf{u}^h||_B ||\mathbf{u}-\mathbf{v}^h||_B, \quad \forall \mathbf{v}^h \in S^h \tag{3.19}$$

We have:

$$||\mathbf{u}-\mathbf{u}^h||_B \le ||\mathbf{u}-\mathbf{v}^h||_B, \quad \mathbf{v}^h \in S^h \tag{3.20}$$

or,

$$||\mathbf{u}-\mathbf{u}^h||_B = \min_{\mathbf{v}^h \in S^h} ||\mathbf{u}-\mathbf{v}^h||_B \tag{3.21}$$

In the following we'll comit a "variational crime", considering the following weak form of the problem:

$$B\left(\mathbf{w}^h, \mathbf{u}^h\right) = l^h\left(\mathbf{w}^h\right), \quad \forall \mathbf{w}^h \in S^h \tag{3.22}$$

where $l\left(\mathbf{w}^h\right)$ is numerically approximated by $l^h\left(\mathbf{w}^h\right)$. In this case Galerkin's orthogonality relation doesn't hold:

$$B(\mathbf{u}-\mathbf{u}^h, \mathbf{w}^h) = l\left(\mathbf{w}^h\right) - l^h\left(\mathbf{w}^h\right) \ne 0, \quad \forall \mathbf{w}^h \in S^h \tag{3.23}$$

and Céa's lemma becomes: if $\mathbf{u} \in S$ and $\mathbf{u}^h \in S^h$ then:

$$||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} \le \frac{k_1\sqrt{2}}{k} \min_{\mathbf{v}^h \in S^h} ||\mathbf{u}-\mathbf{v}^h||_{H^1(D)} + \frac{\sqrt{2}}{k} \sup_{\mathbf{w}^h \in S^h} \frac{|l(\mathbf{w}^h) - l^h(\mathbf{w}^h)|}{||\mathbf{w}^h||_{H^1(D)}} \tag{3.24}$$

From (3.10), we have:

$$\begin{aligned} k||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}^2 &\le B\left(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h\right) \\ &= B\left(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h\right) + B\left(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h-\mathbf{u}^h\right) \\ &\le B\left(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h\right) + l\left(\mathbf{v}^h-\mathbf{u}^h\right) + l^h\left(\mathbf{v}^h-\mathbf{u}^h\right) \\ &\le k_1||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}||\mathbf{u}-\mathbf{v}^h||_{H^1(D)} \\ &\quad + \sup_{\mathbf{w}^h \in S^h} \frac{|l(\mathbf{w}^h) - l^h(\mathbf{w}^h)|}{||\mathbf{w}^h||_{H^1(D)}} ||\mathbf{v}^h-\mathbf{u}^h||_{H^1(D)} \end{aligned} \tag{3.25}$$

Let's denote:

$$||l-l^h||_{-1,h} = \sup_{\mathbf{w}^h \in S^h} \frac{|l(\mathbf{w}^h) - l^h(\mathbf{w}^h)|}{||\mathbf{w}^h||_{H^1(D)}} \tag{3.26}$$

From the triangular inequality and (3.26), the formula (3.25) we have:

$$\begin{aligned} k||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}^2 &\le k_1||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}||\mathbf{u}-\mathbf{v}^h||_{H^1(D)} \\ &\quad + ||l-l^h||_{-1,h}\left(||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} + ||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}\right) \\ &= ||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}\left(k_1||\mathbf{u}-\mathbf{v}^h||_{H^1(D)} + ||l-l^h||_{-1,h}\right) \\ &\quad + ||l-l^h||_{-1,h}||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} \end{aligned} \tag{3.27}$$

More, if we'll consider the inequality:

$$\alpha\beta \le \frac{1}{2k}\alpha^2 + \frac{k}{2}\beta^2, \quad \alpha, \beta \ge 0 \tag{3.28}$$

we obtain:

$$||\mathbf{u}-\mathbf{u}^h||_{H^1(D)}\left(k_1||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+||l-l^h||_{-1,h}\right)$$
$$\leq \frac{1}{2k}\left(k_1||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+||l-l^h||_{-1,h}\right)^2+\frac{k}{2}||\mathbf{u}-\mathbf{u}^h||^2_{H^1(D)} \quad (3.29)$$

In this way, from (3.27) and (3.29) we have:

$$k^2||\mathbf{u}-\mathbf{u}^h||^2_{H^1(D)} \leq \left(k_1||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+||l-l^h||_{-1,h}\right)^2$$
$$+2k||l-l^h||_{-1,h}||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} \quad (3.30)$$

or,

$$k^2||\mathbf{u}-\mathbf{u}^h||^2_{H^1(D)} \leq 2\left(k_1||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+||l-l^h||_{-1,h}\right)^2 \quad (3.31)$$

In other words, we have :

$$k||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} \leq \frac{k_1\sqrt{2}}{k}||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+\frac{\sqrt{2}}{k}||l-l^h||_{-1,h} \quad (3.32)$$

or

$$||\mathbf{u}-\mathbf{u}^h||_{H^1(D)} \leq \frac{k_1\sqrt{2}}{k}\min_{\mathbf{v}^h\in S^h}||\mathbf{u}-\mathbf{v}^h||_{H^1(D)}+\frac{\sqrt{2}}{k}\sup_{\mathbf{w}^h\in S^h}\frac{|l(\mathbf{w}^h)-l^h(\mathbf{w}^h)|}{||\mathbf{w}^h||_{H^1(D)}} \quad (3.33)$$

which is the desired result.

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