LIBERTAS MATHEMATICA vol XXIV (2004)

Exponential Stability for a Class of Neutrals Functional Differential Equations with Finite Delays

Sânziana CARAMAN

Abstract. In the present work we study the asymptotic behaviour of certain neutral retarded functional differential equations in Banach spaces within the framework of semigroup theory and linear operators results.

AMS Subject Classification: 34K06, 34K20, 34K40.

1 Introduction

In this paper we are concerned with the stability of the following delay equation, considered in [1] by R. Datko.

$$\frac{d}{dt}\left[u(t)-\sum_{j=1}^{m}B_ju(t-h_j)\right]=Au(t)+\sum_{j=1}^{m}A_ju(t-h_j),\quad t\geq 0 \tag{1.1}$$

where A is the infinitesimal generator of a C_0–semigroup on a Banach space X, A_j and B_j are linear bounded operators on X and h_j are fixed positive constants.

Using semigroup methods under some hypotheses on the operators $\{A_j\}_{j=1}^{m}$ and $\{B_j\}_{j=1}^{m}$, R. Datko has proved the existence and the uniqueness of the solution for problem (1.1). In [2] he also studied the exponential stability of a similar delay equation on $X=\mathbb{R}^n$.

$$\frac{d}{dt}\left[u(t)-\sum_{j=1}^{m}B_ju(t-\alpha h_j)\right]=\sum_{j=1}^{m}A_ju(t-\alpha h_j),\quad \alpha\geq 0 \tag{1.2}$$

The problem raisen by R. Datko is in what extent do the stability properties of (1.2) depend on a change of α.

We are going, now, to investigate the stability of problem (1.1) in a quite different manner, regarding A_j and B_j as a perturbation of an appropriate Cauchy problem without delays and under some additional hypotheses for A, A_j and B_j, establish sufficient conditions for the exponential stability of the solution.

This technique is already practiced for some time and provides an elegant tool for the study of delay equations (see, for instance, Yan van Neerven [4], whose work generously influenced our present paper, and also S. Nakagiri [3]) but we are interested in extending it to neutral functional differential equations.

2 Preliminaries

Let X be a Banach space with the norm $\|\cdot\|$ and $\mathbb{C}$ the complex plane.

Let $\mathcal{C} = \mathcal{C}_{[[-h,0],X]} = \{\phi \mid \phi : [-h,0] \to X$ is a continuous function$\}$ with the usual sup–norm

$$\|\phi\|_{\mathcal{C}} = \sup\{\|\phi(t)\|, \quad t \in [-h,0]\}, \ \phi \in \mathcal{C}.$$

If X is a Banach space then $\mathcal{L}(X)$ denotes the Banach space of linear bounded operators from X into itself with the sup–norm $\|\cdot\|_{\mathcal{L}(X)}$.

Let $A : D(A) \subset X \to X$ be the generator of a C_0–semigroup of operators on X, $\mathcal{T} = (T(t))_{t\geq 0}$.

Let $\{A_j\}_{j=1}^m$ and $\{B_j\}_{j=1}^m$ be constant operators from $\mathcal{L}(X)$ such that:

$$\operatorname{Range} B_j \subset D(A), \quad AB_j \in \mathcal{L}(X) \quad \text{for every} \quad j = \overline{1,m}$$

and $h_j \in \mathbb{R}_+^*$ with $0 < h_1 < h_2 < \cdots < h_m = h$. Consider the equations:

$$\frac{d}{dt}\left[u(t) - \sum_{j=1}^m B_j u(t-h_j)\right] = Au(t) + \sum_{j=1}^m A_j u(t-h_j), \quad t \geq 0 \tag{2.1a}$$

$$u(t) = \phi(t), \quad t \in [-h,0], \ \phi \in \mathcal{C}. \tag{2.1b}$$

The problem (2.1a)–(2.1b) can be written in the integral form:

$$\begin{aligned} u(t,0,\phi) &= \sum_{j=1}^m B_j(t-h_j;0,\phi) + T(t)\left[\phi(0) - \sum_{j=1}^m B_j\phi(-h_j)\right] + \\ &+ \int_0^t T(t-\tau)\left[\sum_{j=1}^m (A_j + AB_j)u(\tau - h_j, 0, \phi)\right] d\tau, \quad t \geq 0 \end{aligned} \tag{2.2a}$$

$$u(t,0,\phi) = \phi(t), \quad t \in [-h,0], \ \phi \in \mathcal{C}. \tag{2.2b}$$

Every continuous solution of problem (2.2a)–(2.2b) is called a "mild" solution for the problem (2.1a)–(2.1b).

In [1] the author established that for each $\phi \in \mathcal{C}^1$ there exists a unique solution for (2.2a)–(2.2b), which is continuous on $[0,+\infty)$.

Let $u(t,0,\phi)$ be this solution.

In order to study the asymptotic behavior of $u(t,0,\phi)$ by semigroup methods we introduce the application:

$$u_t(\phi,0) = \{u(t+\tau,0,\phi), \quad \tau \in [-h,0]\}$$

and the map:

$$S : \mathbb{R}_+ \to \mathcal{L}(\mathcal{C})$$

defined by:

$$S(t)\phi = u_t(\phi,0) = \{u(t+\tau,0,\phi), \ \tau \in [-h,0]\} \tag{2.3}$$

Proposition 2.1. ([1]) *The map defined by (2.3) is a C_0−semigroup on $\mathcal{C}$. Its generator $\mathcal{A}$ is given by:*

$$D(\mathcal{A}) = \Big\{\phi \in \mathcal{C} \mid \exists \phi' \in \mathcal{C},\ \phi(0) \in D(A) \text{ and } \phi'(0) = \sum_{j=1}^{m} B_j \phi'(-h_j) + \sum_{j=1}^{m} A_j \phi(-h_j) + A\phi(0)\Big\} \tag{2.4}$$

and

$$\mathcal{A}(\phi)(\tau) = \begin{cases} \phi'(\tau) \text{ for } \tau \in [-h, 0) \\ A\phi(0) + \sum_{j=1}^{m} A_j \phi(-h_j) + \sum_{j=1}^{m} B_j \phi'(-h_j) \end{cases} \tag{2.5}$$

In the following, let us consider two constants [5]:

- *the abscissa of uniform boundedness $s_0(A)$ of the resolvent of A*

$$s_0(A) = \inf\{\omega \in \mathbb{R} \mid \{\text{Re } \lambda > \omega\} \subset \rho(A) \text{ and } \sup_{\text{Re } \lambda > \omega} \|R(\lambda, A)\| < \infty\}$$

- *the growth bound $\omega_1(A)$*
  $\omega_1(A) = \inf\{\omega \in \mathbb{R}\}$ *there exists $M \geq 1$ such that*

$$\|T(t)x\| \leq M e^{\omega t} \|x\|_{D(A)} \quad \text{for all } x \in D(A) \text{ and } t \geq 0\}.$$

The relation between them is given in [5]

$$-\infty \leq \omega_1(A) \leq s_0(A) < \infty \tag{2.6}$$

$\mathcal{T} = (T(t)))_{t\geq 0}$ is said to be exponentially stable if $\omega_1(A) < 0$.

3 The stability theorem

Let us, first, remind the following result:

Lemma 3.1. ([5]) *If A is a closed operator on a Banach space X and $\Omega \subset \rho(A)$ such that $\sup_{\lambda\in\Omega} \|R(\lambda, A)\| \leq M$ and if $\Omega_\delta = \{\lambda \in \mathbb{C} \mid \text{dist}(\lambda, \Omega) \leq \delta\}$, where $\delta = \frac{1}{2M}$, then $\Omega_\delta \subset \rho(A)$ and $\sup_{\lambda\in\Omega_\delta} \|R(\lambda, A)\| \leq 2M$.*

Theorem 3.2. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a C_0−semigroup of linear operators on X and $A : D(A) \subset X \to X$ its infinitesimal generator; suppose $s_0(A) < 0$ and consider the delay equation*

$$\begin{cases} \dfrac{d}{dt}\left(u(t) - \sum_{j=1}^{m} B_j u(t - h_j)\right) = Au(t) + \sum_{j=1}^{m} A_j u(t - h_j) \\ u(t) = \phi(t), \quad t \in [-h, 0],\ \phi \in \mathcal{C} \end{cases} \tag{3.1}$$

where $A_j, B_j \in \mathcal{L}(X)$, *Range* $B_j \subset D(A)$ *and* $AB_j \in \mathcal{L}(X)$ *for every* $j = \overline{1,m}$ *and* $h_j \in \mathbb{R}_+^*$ *with* $0 < h_1 < h_2 < \cdots < h_m = h$. *If,*

$$\sup_{\omega\in\mathbb{R}} \left\| R\left(i\omega, A\sum_{j=1}^{m} e^{-i\omega h_j}\left(A_j + i\omega B_j\right)\right)\right\| < 1, \tag{3.2}$$

then the solutions of (3.1) *are exponentially stable.*

Proof. First we prove that under the above hypotheses,

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \subset \rho\left(A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right). \qquad *$$

As $s_0(A) < 0$ we have that $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \subset \rho(A)$ and $\sup\limits_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| < \infty$. Fix $\lambda \in \rho(A)$, with $\operatorname{Re}\lambda > 0$ and define:

$$V(\lambda) : X \to X,\ V(\lambda) = R(\lambda, A)\sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)$$

Let us observe that the function:

$$V : \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \to \mathcal{L}(X)$$

is a bounded analytic function because:

$$\begin{aligned} V(\lambda) &= R(\lambda, A)\sum_{j=1}^{m} e^{-\lambda h_j} A_j + \sum_{j=1}^{m} e^{-\lambda h_j} R(\lambda, A)\lambda B_j = \\ &= R(\lambda, A)\sum_{j=1}^{m} e^{-\lambda h_j} A_j + \sum_{j=1}^{m} e^{-\lambda h_j}(R(\lambda, A)AB_j + B_j) = \\ &= \sum_{j=1}^{m} e^{-\lambda h_j}[R(\lambda, A)(A_j + AB_j) + B_j] \end{aligned}$$

$$\|V(\lambda)\| \le \sum_{j=1}^{m}(\|R(\lambda, A)\|(\|A_j\| + \|AB_j\|) + \|B_j\|)$$

Since A_j, B_j, and $AB_j \in \mathcal{L}(X)$ for every $j = \overline{1,m}$ and for $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$ also $R(\lambda, A)$ is bounded, it follows that $V(\lambda)$ is also bounded.

As the suprema along vertical lines $\operatorname{Re}\lambda = a$ of bounded analytic functions decreases as a increases, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, and using (3.2) we have:

$$\|V(\lambda)\| \le \sup_{\omega\in\mathbb{R}} \|V(i\omega)\| \le 1 - \delta, \quad \text{for } \delta \in (0, 1) \tag{3.3}$$

It follows that $I - V(\lambda)$ is invertible and by Newmann series we obtain:

$$\left\|(I - V(\lambda))^{-1}\right\| \le \sum_{n=0}^{\infty}(1 - \delta)^n = \frac{1}{\delta} \tag{3.4}$$

Let us denote by $\Delta(\lambda) : D(A) \subset X \to X$ the operator:

$$\Delta(\lambda) = \lambda I - A - \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j) \text{ with } \lambda \in \rho(A) \text{ and } \operatorname{Re}\lambda > 0.$$

If we use the identity:

$$\Delta(\lambda) = (\lambda I - A)(I - V(\lambda)), \tag{3.5}$$

it follows that $\Delta(\lambda)$ is a closed, invertible operator, being the composition of a bounded, invertible operator and a closed, invertible one. Therefore, for $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$, it exists $\Delta^{-1}(\lambda) : X \to X$ and by the closed graph theorem it is a closed operator,

$$\Delta^{-1}(\lambda) = R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right)$$

and taking into account (3.5) and (3.4) we have:

$$\begin{gathered} R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right) = (I - V(\lambda))^{-1} R(\lambda, A) \text{ and} \\ \left\| R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right) \right\| \le \frac{1}{\delta}\|R(\lambda, A)\| \end{gathered} \tag{3.6}$$

The following step is to prove that

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\} \subset \rho(\mathcal{A}) \tag{**}$$

where $\mathcal{A}$ is defined as in Section 2.

Let $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$ and let $\varphi \in \mathcal{C}$. We construct $g \in D(\mathcal{A})$ so that:

$$(\lambda I - \mathcal{A})g = \varphi \tag{3.7}$$

where:

$$D(\mathcal{A}) = \left\{ g \in \mathcal{C} \mid g' \in \mathcal{C},\ g(0) \in D(A),\ g'(0) = \sum_{j=1}^{m} B_j g'(-h_j) + \sum_{j=1}^{m} A_j g(-h_j) + Ag(0) \right\}$$

and

$$(\mathcal{A}g)(s) = \begin{cases} g'(s);\ s \in [-h, 0) \\ Ag(0) + \sum_{j=1}^{m} A_j g(-h_j) + \sum_{j=1}^{m} B_j g'(-h_j) \end{cases} \tag{3.8}$$

By (3.7) and (3.8) we have:

$$g'(s) - \lambda g(s) = -\varphi(s) \text{ for } s \in [-h, 0) \tag{3.9}$$

and

$$\lambda g(0) - Ag(0) - \sum_{j=1}^{m} A_j g(-h_j) - \sum_{j=1}^{m} B_j g'(-h_j) = \varphi(0) \tag{3.10}$$

Solving (3.9), we obtain:

$$g(s) = e^{\lambda s} g(0) + \int_s^0 e^{\lambda(s-\zeta)} \varphi(\zeta) d\zeta, \ s \in [-h, 0) \tag{3.11}$$

And by (3.10), (3.11):

$$\lambda g(0) - A g(0) - \sum_{j=1}^{m} A_j \left(e^{-\lambda h_j} g(0) + \int_{-h_j}^{0} e^{\lambda(-h_j-\zeta)} \varphi(\zeta) d\zeta \right) -$$

$$- \sum_{j=1}^{m} B_j \left[\lambda e^{-\lambda h_j} g(0) + \lambda \int_{-h_j}^{0} e^{\lambda(-h_j-\zeta)} \varphi(\zeta) d\zeta - \varphi(-h_j) \right] = \varphi(0)$$

And finally:

$$\Delta(\lambda) g(0) = \sum_{j=1}^{m} e^{-\lambda h_j} (A_j + \lambda B_j) \int_{-h_j}^{0} e^{-\lambda\zeta} \varphi(\zeta) d\zeta - \sum_{j=1}^{m} B_j \varphi(-h_j) + \varphi(0)$$

As $\lambda \in \rho \left(A + \sum\limits_{j=1}^{m} e^{-\lambda h_j} (A_j + \lambda B_j) \right)$ it follows that:

$$g(0) = R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j} (A_j + \lambda B_j) \right) (S_\lambda \varphi + \varphi(0)) \tag{3.12}$$

and

$$g = E_\lambda R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j} (A_j + \lambda B_j) \right) (S_\lambda \varphi + \varphi(0)) + T_\lambda \varphi \tag{3.13}$$

where:

$$E_\lambda : X \to \mathcal{C}, \quad E_\lambda x = e^{\lambda \cdot} x$$

$$S_\lambda : \mathcal{C} \to X, \quad S_\lambda \varphi = \sum_{j=1}^{m} e^{-\lambda h_j} (A_j + \lambda B_j) \int_{-h_j}^{0} e^{-\lambda\zeta} \varphi(\zeta) d\zeta - \sum_{j=1}^{m} B_j \varphi(-h_j)$$

$$T_\lambda : \mathcal{C} \to \mathcal{C}, \quad T_\lambda \varphi = \int_{\cdot}^{0} e^{\lambda(\cdot - \zeta)} \varphi(\zeta) d\zeta$$

Observe first that:

$$\|E_\lambda x\|_C \le \|E_\lambda\|_{\mathcal{L}(X,C)} \|x\|$$

$$\|E_\lambda x\|_C = \left\|e^{\lambda \cdot} x\right\|_C = \sup_{s \in [-h,0]} \left\|e^{\lambda s} x\right\| = \sup_{s \in [-h,0]} \left|e^{\lambda s}\right| \|x\| = \|x\|$$

Thus

$$\|E_\lambda\|_{\mathcal{L}(X,C)} = 1$$

$$\|T_\lambda \varphi\|_C \le \|T_\lambda\|_{\mathcal{L}(C)} \|\varphi\|_C$$

$$\|T_\lambda \varphi\|_C = \sup_{s \in [-h,0]} \left\| \int_s^0 e^{\lambda(s-\zeta)} \varphi(\zeta) d\zeta \right\| \le \|\varphi\|_C \sup_{s \in [-h,0]} \int_s^0 e^{\mathrm{Re}\ \lambda(s-\zeta)} d\zeta =$$

$$= \|\varphi\|_C \sup_{s \in [-h,0]} \left| \frac{e^{(\mathrm{Re}\ \lambda)s} - 1}{(\mathrm{Re}\ \lambda)s} \right| \cdot |-s| = h \|\varphi\|_C$$

Therefore $\|T_\lambda\|_{\mathcal{L}(C)} \leq h$.

It remains to study the boundedness of

$$R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right) S_\lambda \varphi. \tag{***}$$

By (3.6) and the identity

$$R(\lambda, A)\lambda B_j = R(\lambda, A)AB_j + B_j$$

we obtain:

$$\begin{aligned} &R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right) S_\lambda \varphi = \\ &= (I - V(\lambda))^{-1} R(\lambda, A) \sum_{j=1}^{m} e^{-\lambda h_j} A_j \int_{-h_j}^{0} e^{-\lambda h_j} \varphi(\zeta) d\zeta + \\ &+ (I - V(\lambda))^{-1} \sum_{j=1}^{m} e^{-\lambda h_j} (R(\lambda, A)AB_j + B_j) \int_{-h_j}^{0} e^{-\lambda h_j} \varphi(\zeta) d\zeta - \\ &- R\left(\lambda, A + \sum_{j=1}^{m} e^{-\lambda h_j}(A_j + \lambda B_j)\right) \sum_{j=1}^{m} B_j \varphi(-h_j) \end{aligned}$$

Therefore, if $\lambda \in \rho(A)$ with Re $\lambda > 0$ we have:

$$\begin{aligned} \|g\|_C \leq \Bigg(&\frac{1}{\delta}\|R(\lambda, A)\| \sum_{j=1}^{m} \|A_j\| h + \frac{1}{\delta} \sum_{j=1}^{m} (\|R(\lambda, A)\| \|AB_j\| + \|B_j\|) h + \\ &+ \frac{1}{\delta}\|R(\lambda, A)\| \sum_{j=1}^{m} \|B_j\| + h \Bigg) \|\varphi\|_C \end{aligned}$$

As A_j, B_j and $AB_j \in \mathcal{L}(X)$ for $j = \overline{1, m}$ and $R(\lambda, A)$ is bounded, it follows that there exists $k \in R_+^*$, not depending on λ, such that:

$$\frac{h}{\delta}\|R(\lambda, A)\| \sum_{j=1}^{m} \left(\|A_j\| + \|AB_j\| + \left(1 + \frac{1}{h}\right) \|B_j\| \right) + h \leq k < \infty$$

or $\|g\|_C \leq k\|\varphi\|_C$, thus $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > 0\} \subset \rho(\mathcal{A})$ and as $\|g\|_C = \|R(\lambda, \mathcal{A})\varphi\|_C \leq \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(C)} \|\varphi\|_C$ it follows that

$$\|R(\lambda, \mathcal{A})\|_{\mathcal{L}(C)} \leq k \text{ and } \sup_{\text{Re } \lambda > 0} \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(C)} < \infty$$

Using now Lemma 3.1, it results that $s_0(\mathcal{A}) < 0$ and, by (2.6) $\omega_1(\mathcal{A}) < 0$, which means that there exists $M > 0$ and $\omega > 0$, such that for all $\phi \in D(\mathcal{A})$ we have:

$$\|u(t, 0, \phi)\| \leq M e^{-\omega t} \|\phi\|_{D(\mathcal{A})}.$$

References

[1] Datko, R., *Linear Autonomous Neutral Differential Equations in Banach space*, Journ. Diff. Eqs. vol. 25, nr.2 (1977), 238–274.

[2] Datko, R., *A procedure for determination of the exponential stability of certain differential–difference equations*, Quarterly of Applied Mathematics Oct. (1978), 277–292.

[3] Nakgiri, S., *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. 25 (1988), 353–398.

[4] van Neerven, Y., *Robust Stability of C_0–Semigroups and an Application to Stability of Delay Equations*, Journ. of Math. Analysis and Appl. 226 (1998), 82–100.

[5] van Neerven, Y., *The asymptotic behaviour of semigroups of linear operator in "Operator Theory": Advances and Applications*, vol. 88, Birkhäuser Verlag, Basel Boston/ Berlin 1996.