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Harmonic Semi-Riemannian Structures with Potential

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Abstract. The notion of harmonic semi-Riemannian structure with potential is introduced here as an extension of the harmonic Riemannian structure defined in [1]. We then provide some properties and an example.

Keywords: semi-Riemannian manifolds, harmonic maps, harmonic maps with potential.

Harmonic maps is a very important topic in Differential Geometry and Analysis. However, some maps between Riemannian manifolds are not harmonic, but they are harmonic with potential.

This is why, in the last few years, a lot of papers extended from the notion of harmonicity to the notion of harmonicity with potential. In what follows we provide such an extension: that is we give here a generalization of a notion introduced in [1] by Chen and Nagano.

If $\varphi : (M,g) \to (N,h)$ is a map between semi-Riemannian manifolds, with ∇ and ∇^h the corresponding Levi - Civita connection, then:

$$\tau^{\lambda}(\varphi) = \tau(\varphi) + \operatorname{grad}\lambda, \quad \forall \lambda \in \mathcal{F}(N), \tag{1}$$

where $\tau(\varphi)$ is the tension of φ defined as the trace of the second fundamental form:

$$\nabla d\varphi(X,Y) = \overset{h}{\nabla}_{d\varphi X} d\varphi Y - d\varphi(\nabla_X Y), \quad \forall X, Y \in \Gamma(TM). \tag{2}$$

The map φ *is called harmonic* (resp. *harmonic with the potential* λ) if $\tau = 0$ (resp. $\tau^{\lambda} = 0$) [2] - [7].

In what follows, we take (M,g) to be a semi-Riemannian manifold and let h be any semi-Riemannian metric on M, with ∇ and $\overset{h}{\nabla}$ the corresponding Levi - Civita connections.

In [1] h is called *harmonic* (with respect to g) provided the identity map $1_M : (M,g) \to (M,h)$ is harmonic. It is then natural to introduce here the following:

Definition 1. *On a semi-Riemannian manifold* (M,g)*, any semi-Riemannian structure* h *is harmonic with the potential* λ (with respect to g) *if* $1_M : (M,g) \to (M,h)$ *is a harmonic* *map with potential* $\lambda \in \mathcal{F}(M)$.

Formula. For any $X \in \Gamma(TM)$ and $\lambda \in \mathcal{F}(M)$ we have:

$$\operatorname{trace} \nabla_X h = 2\left\{\operatorname{div} h(X) + h(X, \operatorname{grad}\lambda) - h\left(X, \tau^\lambda\right)\right\}. \tag{3}$$

Proof. Let (x^i) be any local coordinates on M. Then

$$\begin{aligned}
\operatorname{div} h(\partial_j) &= g^{ab}(\nabla_a h)(\partial_j, \partial_b) = g^{ab}\left\{\partial_a h(\partial_j, \partial_b) - h(\nabla_a \partial_j, \partial_b) - h(\partial_j, \nabla_a \partial_b)\right\} = \\
&= g^{ab}\left\{h\left(\overset{h}{\nabla}_a \partial_j, \partial_b\right) + h\left(\partial_j, \overset{h}{\nabla}_a \partial_b\right) - h(\nabla_a \partial_j, \partial_b) - h(\partial_j, \nabla_a \partial_b)\right\} = \\
&\quad \text{(from (1) and the symmetry of } \overset{h}{\nabla}_a \partial_j \text{ and } \nabla_a \partial_j) \\
&= g^{ab}\left\{h\left(\overset{h}{\nabla}_j \partial_a, \partial_b\right) - h(\nabla_j \partial_a, \partial_b)\right\} + h\left(\partial_j, \tau^\lambda(1_M) - \operatorname{grad}\lambda\right).
\end{aligned}$$

Since $\overset{h}{\nabla}$ is the Levi - Civita connection of h, we obtain:

$$\operatorname{trace} \nabla_i h = 2\{\operatorname{div} h(\partial_i) + h(\partial_i, \operatorname{grad}\lambda) - h\left(\partial_i, \tau^\lambda(1_M)\right)\}, \quad i = \overline{1, m}, \tag{4}$$

which is equivalent to (3) and complete the proof.

Let $G, H : TM \to T^*M$ be the bundle isomorphisms defined by g and h respectively.

The formula (3) yields the following equivalences:

Proposition 1. *h is harmonic with potential* $\Leftrightarrow \exists \lambda \in \mathcal{F}(M)$ *such that*

$$\operatorname{trace} \nabla_X h = 2\left\{\operatorname{div} h(X) + h(X, \operatorname{grad}\lambda)\right\} \Leftrightarrow \exists \lambda \in \mathcal{F}(M) \tag{5}$$

such that

$$\operatorname{trace} \nabla h = 2\left\{\operatorname{div} h + \operatorname{div} \lambda h - \lambda \operatorname{div} h\right\} \Leftrightarrow \tag{6}$$

$\Leftrightarrow$ *the form* $GH^{-1}\{\operatorname{trace} \nabla h - 2\operatorname{div} h\}$ *is exact, where* $\operatorname{trace} \nabla h : X \in \Gamma(TM) \mapsto \operatorname{trace}(\nabla_X h) \in \mathcal{F}(M)$ *is viewed as a 1-form.*

Remark. In particular, h is harmonic $\Leftrightarrow \operatorname{trace} \nabla h = 2\operatorname{div} h$.

Corollary 2. *Assume the first Betti number is zero. Then h is harmonic with potential if and only if the operation*

$$A : X \in \Gamma(TM) \mapsto \nabla_X H^{-1}\{\operatorname{trace} \nabla h - 2\operatorname{div} h\} \in \Gamma(TM)$$

is symmetric with respect to g.

Proof. From a straightforward calculation, the symmetry of A with respect to g is equivalent to the closedness of the form $GH^{-1}\{\operatorname{trace} \nabla h - 2\operatorname{div} h\}$. Now the Corollary follows from the vanishing of the first Betti number.

Definition 1 can be extended as follows:

Definition 2. *On a semi-Riemannian manifold* (M, g), *any symmetric* $(0,2)$*-tensor field is harmonic with potential* $\lambda \in \mathcal{F}(M)$ (with respect to g), *provided* h *satisfies* (5).

Proposition 3. *On a compact Riemannian manifold* (M, g), *let the vector space of all symmetric* $(0,2)$*-tensor fields be endowed with the standard inner product. Then any symmetric* $(0,2)$*-tensor field* h *is harmonic with potential* λ *if and only if there exists a sequence of semi-Riemannian metrics harmonic with potential* λ *convergent to* h.

Proof. Let h be a symmetric $(0,2)$-tensor field. If g_n, $n \in \mathbb{N}$ is a sequence of semi-Riemannian metrics harmonic with potential which is convergent to h, then g_n satisfies (5) and yield to the limit that h satisfies (5) as well. Conversely, let h be harmonic with potential λ. If h is non-degenerate, the wished sequence is the constant one $g_n = h, \forall n \in \mathbb{N}$. Otherwise, from (5), $g_n = h - \frac{1}{n}g$ is harmonic with potential λ for any $n \in \mathbb{N}$ and obviously, this sequence converges to h. In a local chart U of each $p \in M$, we chose an orthonormal basis of g and express g_n as a matrix $H - \frac{1}{n}I$, $n \in \mathbb{N}$ (where I denotes the identity matrix). Since H has at most m $(m = \dim M)$ eigenvalues in p, then $H - \frac{1}{n}I$ is non-singular on an open nieghbourhood $V_p \subset U$ of p for n sufficiently large, say $n \geq n_p$. As M is compact, it has a finite covering $V_{s_1}, \dots, V_{s_k}$ and hence g_n is non-degenerate on M for $n \geq \max\{n_{s_1}, \dots, n_{s_k}\}$. Therefore g_n is the wished sequence which complete the proof.

Remark. In particular, Proposition 3 rewrites for harmonicity without potential.

From (5) we obtain:

Proposition 4. *Let* (M, g) *be a compact Riemannian manifold. Then in the vector space of all symmetric* $(0,2)$*-tensor fields endowed with the standard inner product, those which are harmonic with potential form a closed vector subspace.*

Example. Let S^1 be the unit sphere, endowed with the standard Riemannian metric g and let $h = \gamma g$ be another Riemannian structure on S^1, with γ a positive function. Then it is easily to verify that h is a harmonic Riemannian structure with the potential $\lambda = -\ln \gamma$.

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