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Cohomology of Fields of 2-jets on a Foliated Manifold

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Abstract. Let M be a Riemannian foliated manifold. We define the fields of leafwise and transversal 2-jets on M. We obtain a relation between the basic cohomology of M and the cohomology of fields of basic 2-jets. A relation between the cohomology of fields of leafwise 2-jets and the leafwise cohomology of M is also obtained.

1 Preliminaries

Following [2], [3] we present some notions containing the fields of 2-jets on a manifold.

Let M be a n-dimensional manifold and denote by $\Omega^0(M)$ the set of differential functions on M.

We recall that two maps $f, g \in \Omega^0(M)$ determine the same *2-jet at* $x \in M$ if $f(x) = g(x) = 0$ and if for every curve $\gamma : R \to M$ with $\gamma(0) = x$, the curves $f \circ \gamma$, $g \circ \gamma$ have a second order contact at zero. If, moreover, the maps f and g have the first derivative in x equal to zero, then they determine the same *homogeneous 2-jet at* x.

The *2-jet of* f *at* x is denoted by $j_x^2 f$. Obviously, $j_x^2 f$ depends on the germ of f at x only.

Let (U, φ), $(\bar{U}, \bar{\varphi})$ be two local charts with $(x^1, ..., x^n)$, $(\bar{x}^1, ..., \bar{x}^n)$ as local coordinates, respectively.

More generally a *2-jet at* $x \in M$ is a combination locally given by

$$\omega_x^2 = \omega_i(x) \cdot j_x^2 x^i + \frac{1}{2}\omega_{ij}(x) \cdot j_x^2 x^i \cdot j_x^2 x^j \tag{1.1}$$

with $\omega_{ij} = \omega_{ji}$ and where the coefficient functions are satisfying the following conditions in $U \cap \bar{U}$

$$\bar{\omega}_i = \omega_j \frac{\partial x^j}{\partial \bar{x}^i}; \quad \bar{\omega}_{i_1 i_2} = \omega_{j_1 j_2} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} + \omega_j \frac{\partial^2 x^j}{\partial \bar{x}^{i_1} \partial \bar{x}^{i_2}} \tag{1.2}$$

The set of all 2-jets at x is denoted by $J_x^2 M$. The set $J^2 M = \bigcup\limits_{x \in M} J_x^2(M)$ of 2-jets on M is a fiber bundles over M with the fiber of dimension $\dfrac{n(n+3)}{2}$ and we denote by $J^2(M)$

its sections, namely the space of fields of 2-jets on M.

Let $j^2 : \Omega^0(M) \longrightarrow J^2(M)$ be the map who assigns to a function f the field $j^2 f$ of 2-jets of f on M. This map is called the *second differential* on M. In local coordinates we have $j^2 f = \frac{\partial f}{\partial x^i} j^2 x^i + \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j} j^2 x^i \cdot j^2 x^j$.

The field of 2-jets $\omega \in J^2(M)$ is called [3]:

(i) *exact* if $\omega = j^2 f$ for some $f \in \Omega^0(M)$;

(ii) *closed* if it is locally exact.

We denote by $E^2(M)$, $C^2(M)$ the space of exact, respectively closed fields of 2-jets. We have $E^2(M) \subset C^2(M)$ and the *cohomology group of fields of 2-jets on* M is the following quotient group:

$$H_j^2(M) = C^2(M)/E^2(M) \tag{1.3}$$

It is isomorphic with the 1-dimensional cohomology group with real coefficients of M ([3], theorem II).

Now, we present some notions about the foliated manifolds using [1],[7],[9]. In this paper we consider that indices take the following values: $a, b, ... = 1, ..., n$; $u, v, ... = n+1, ..., n+m$; $i, j, ... = 1, ..., n+m$.

Let M be a $n+m$ dimensional manifold. A foliation $\mathcal{F}$ of dimension m and codimension n on M is a partition $\{\mathcal{L}_\alpha\}_{\alpha \in I}$ of M into connected m-dimensional submanifolds called *leaves* and which have the following property. For every point of M there is a local chart with domain U and local coordinates $(x^a, x^u)_{a,u}$, such that for each leaf $\mathcal{L}_\alpha$ the connected components of the *plaque* $U \cap \mathcal{L}_\alpha$ are defined by the equations $dx^a = 0$, for all $a = \overline{1,n}$. Such a chart is called a *distinguished chart*.

The bundle of vectors tangent to leaves is the structural bundle of $\mathcal{F}$. It will be denoted by $T\mathcal{F}$ and its sections by $\mathfrak{X}(\mathcal{F})$. If g is a Riemannian metric on M, the transversal bundle of foliation is the normal subbundle of the structural bundle with respect to g and it will be denoted by $T\mathcal{F}^\perp$; the set of its sections is $\mathfrak{X}^\perp(\mathcal{F})$.

In a distinguished chart $(x^a, x^u)_{\substack{a=1,...n \\ u=n+1,..,n+m}}$ there is an adapted basis

$$\left\{ X_a = \frac{\partial}{\partial x^a} - t_a^u \frac{\partial}{\partial x^u}, \frac{\partial}{\partial x^u} \right\} \tag{1.4}$$

in the module of vector fields of M such that $T\mathcal{F} = span\left(\frac{\partial}{\partial x^u}\right); T\mathcal{F}^\perp = span\,(X_a)$. The functions t_a^u are complete determined by the orthogonality of $T\mathcal{F}$ and $T\mathcal{F}^\perp$. The dual cobasis of the basis $\left\{X_a, \frac{\partial}{\partial x^u}\right\}$ is

$$\{dx^a, \theta^u = dx^u + t_a^u dx^a\} \tag{1.5}$$

We recall that a function $f \in \Omega^0(M)$ is *basic (foliated)* if it is constant on the leaves. We denote by Φ the sheaf of germs of basic functions on M.

Let be $\Omega^{p,q}(M)$ the set of $(p,q)-$ forms and $d_{01} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)$ the *foliated derivative* on M. The *(p,q)-cohomology group* $H^{pq}(M)$ is the q-dimensional cohomology group of the complex $(\Omega^{p,\cdot}(M), d_{01})$. On $\Omega^{0,\cdot}$ the foliated derivative d_{01} can be identified with the operator $d_{\mathcal{F}}$ defined by the de Rham derivative on the leaves.

The *leafwise de Rham complex* $(\Omega^{\cdot}(\mathcal{F}), d_{\mathcal{F}})$ is the restriction to the leaves of the de Rham complex (Ω, d) of M, where $\Omega(\mathcal{F})$ is the set of differential forms on the leaves. The cohomology $H^{\cdot}(\mathcal{F}) = H^{\cdot}(\Omega(\mathcal{F}), d_{\mathcal{F}})$ is called the *leafwise cohomology* of $\mathcal{F}$. Is well-known the following de Rham type theorem:

Theorem 1.1. [1] *$H^q(F)$ and the Čech cohomology group $H^q(M, \Phi)$ with coefficients in the sheaf Φ of germs of basic functions are isomorphic.*

A $(p,0)$-form closed with respect to the foliated derivative is called a *basic (foliated) p-form.* We denote by $\Omega(M/_{\mathcal{F}})$ the space of basic forms. The *basic complex* of M is the subcomplex $(\Omega(M/_{\mathcal{F}}), d)$ of the de Rham complex of M and the cohomology $H^{\cdot}(M/_{\mathcal{F}}) = H^{\cdot}(\Omega(M/_{\mathcal{F}}), d)$ is called the *basic cohomology.*

2 Fields of transversal 2-jets

Let M be a Riemannian foliated manifold like in the previous section. For every $f \in \Omega^0(M)$ we have in a distinguished chart the following decomposition

$$df = (X_a f) dx^a + \frac{\partial f}{\partial x^u} \theta^u \tag{2.1}$$

with respect to the adapted cobasis (1.5).

For two distinguished local charts with domains U, $\bar{U}$ and local coordinates (x^a, x^u), $(\bar{x}^b, \bar{x}^v)$, we obtain by a direct calculation the following relations in $U \cap \bar{U} \neq \emptyset$

$$\bar{X}_b f = \frac{\partial x^a}{\partial \bar{x}^b} X_a f; \;\; \bar{X}_{b_1} \bar{X}_{b_2} f = \frac{\partial^2 x^a}{\partial \bar{x}^{b_1} \partial \bar{x}^{b_2}} X_a f + \frac{\partial x^{a_1}}{\partial \bar{x}^{b_1}} \frac{\partial x^{a_2}}{\partial \bar{x}^{b_2}} X_{a_1} X_{a_2} f$$

We obtain that the functions $X_a f$ and $\frac{1}{2}(X_{a_1} X_{a_2} f + X_{a_2} X_{a_1} f)$ verify the relations (1.2), hence

$$j^{t,2} f = (X_a f) j^2 x^a + \frac{1}{4}(X_{a_1} X_{a_2} f + X_{a_2} X_{a_1} f) j^2 x^{a_1} j^2 x^{a_2} \tag{2.2}$$

is a field of 2-jets on M and we call it the field of *transversal 2-jets of the function f.*

More generally, a field of 2-jets (1.1) given in a distinguished chart by

$$\omega^{t,2} = \omega_a \cdot j^2 x^a + \frac{1}{2} \omega_{a_1 a_2} \cdot j^2 x^{a_1} \cdot j^2 x^{a_2} \tag{2.3}$$

is called a *field of transversal 2-jets on M*. We denote by $J^2(M/_{\mathcal{F}})$ the space of all fields of transversal 2-jets on M.

The map $j^{t,2} : \Omega^0(M) \to J^2(M/_{\mathcal{F}})$ which assign to every differential function f on M its field of transversal 2-jets $j^{t,2} f$ is called the *transversal second differential on M*.

Remark 2.1. Let $(M', \mathcal{F}')$ be another foliated manifold with $\dim \mathcal{F}' = m$, $\operatorname{codim} \mathcal{F}' = n'$ and $\mu : M \to M'$ a foliated map between M and M' (μ send leaves into leaves). This map induces a map $\mu^2 : J^2(M') \to J^2(M)$ locally given by

$$\mu^2(\omega'_i j^2 y^i + \frac{1}{2}\omega'_{ij} j^2 y^i \cdot j^2 y^j) = \omega'_i \frac{\partial y^i}{\partial x^k} j^2 x^k + \frac{1}{2}\left(\omega'_i \frac{\partial^2 y^i}{\partial x^k \partial x^l} + \omega'_{ij}\frac{\partial y^i}{\partial x^k \partial x^l}\right) j^2 x^k j^2 x^l$$

where $(x^1, .., x^{n+m})$, $(y^1, ..., y^{n'+m'})$ are distinguished coordinates of two local charts at x, $\mu(x)$, respectively.

We denote by $\mu^{t,2}$ the restriction of μ^2 to $J^2(M'/_{\mathcal{F}'})$. Taking into account that μ is foliated, we have $\dfrac{\partial y^a}{\partial x^u} = 0$ for all $a = \overline{1, n'}$ and $u = \overline{n+1, n+m}$, so

$$\mu^{t,2}(\omega'_a j^2 y^a + \frac{1}{2}\omega'_{ab} j^2 y^a \cdot j^2 y^b) = \omega'_a \frac{\partial y^a}{\partial x^{a_1}} j^2 x^{a_1} + \frac{1}{2}(\omega'_a \frac{\partial^2 y^a}{\partial x^{a_1}\partial x^{b_1}} + \omega'_{ab}\frac{\partial y^a}{\partial x^{a_1}}\frac{\partial y^b}{\partial x^{b_1}}) j^2 x^{a_1} j^2 x^{b_1}$$

where $a, b = \overline{1, n'}$ and $a_1, b_1 = \overline{1, n}$. The following map is well-defined

$$\mu^{t,2} : J^2(M'/_{\mathcal{F}'}) \to J^2(M/_{\mathcal{F}}) \tag{2.4}$$

If μ is a diffeomorphism, then $\mu^{t,2}$ becomes an isomorphism.

Definition 2.1. *A field of transversal 2-jets (2.3) is called a* ***field of basic 2-jets*** *if all coefficient functions* ω_a, $\omega_{a_1 a_2}$ *are basic functions.*

We shall denote by $J_b^2(M/_{\mathcal{F}})$ the set of fields of basic 2-jets on M and it is a subspace of $J^2(M/_{\mathcal{F}})$. It is easy to see that $\mu^{t,2}(J_b^2(M'/_{\mathcal{F}'}) \subset J_b^2(M/_{\mathcal{F}})$.

Remark 2.2. If f is a basic function, then $j^{t,2}f$ is basic and it has the following form $j^{t,2}f = \dfrac{\partial f}{\partial x^a} j^2 x^a + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^{a_1}\partial x^{a_2}} j^2 x^{a_1} j^2 x^{a_2}$.

Definition 2.2. *The field of basic 2-jets* $\omega \in J_b^2(M/_{\mathcal{F}})$ *is called:*

(i) ***exact*** *if* $\omega = j^{t,2}f$ *for some* $f \in \Phi(M)$*;*

(ii) ***closed*** *if it is locally exact .*

We denote by $E_b^2(M/_{\mathcal{F}})$ and $C_b^2(M/_{\mathcal{F}})$ the spaces of exact, respectively closed fields of basic 2-jets on M and we have $E_b^2(M/_{\mathcal{F}}) \subset C_b^2(M/_{\mathcal{F}})$.

We call the *cohomology group of fields of basic 2-jets on M* the following quotient group:

$$H_j^{b,2}(M/_{\mathcal{F}}) = C_b^2(M/_{\mathcal{F}})/E_b^2(M/_{\mathcal{F}}) \tag{2.5}$$

Now we define a map between the space $\Omega^1(M/_{\mathcal{F}})$ of basic 1-forms on M and the space $J_b^2(M/_{\mathcal{F}})$ of fields of basic 2-jets on M. We show that this map induces an isomorphism between the cohomology group $H_j^{b,2}(M/_{\mathcal{F}})$ defined in (2.5) and the 1-dimensional basic cohomology group $H^1(M/_{\mathcal{F}})$ of M.

Let $\{dx^a, \theta^u\}$ be an adapted cobasis (1.5) on M. To every basic 1-form ω locally given by $\omega = \omega_a dx^a$, we associate

$$D^t\omega = \omega_a \cdot j^2x^a + \frac{1}{4}\left(X_{a_1}\omega_{a_2} + X_{a_2}\omega_{a_1}\right) j^2x^{a_1} \cdot j^2x^{a_2} \tag{2.6}$$

Taking into account the transformation rule for the components of a basic 1–form at the local chart changing, by a direct calculation we obtain that the coefficient functions of $D^t\omega$ satisfy the relations (1.2), so (2.6) is a field of 2-jets which is basic because the functions ω_a are basic. The map $D^t : \Omega^1(M/_{\mathcal{F}}) \longrightarrow J_b^2(M/_{\mathcal{F}})$ defined by (2.6) is a monomorphism.

Theorem 2.1. *Let M be a Riemannian foliated manifold. The cohomology group $H_j^{b,2}(M/_{\mathcal{F}})$ of fields of basic 2-jets on M and the basic cohomology group $H^1(M/_{\mathcal{F}})$ are isomorphic.*

Proof. We denote by $Z^1(M/_{\mathcal{F}})$ and $B^1(M/_{\mathcal{F}})$ the sets of d-closed and d-exact basic 1-forms, respectively. We have the relations:

$$D^t(Z^1(M/_{\mathcal{F}}) = C_b^2(M/_{\mathcal{F}}); \quad D^t(B^1(M/_{\mathcal{F}})) = E_b^2(M/_{\mathcal{F}}) \tag{2.7}$$

Indeed, let be $\theta \in Z^1(M/_{\mathcal{F}})$; from Poincaré lemma for d we have that for an open subset V of M there is $f_V \in \Phi(V)$ such that $\theta\,|_V = df_V = (X_a f_V)dx^a = \dfrac{\partial f_V}{\partial x^a}dx^a$. The last equality is because f is a basic function. Taking into account the definition of the map D^t, we have $D^t\theta\,|_V = j^{t,2}f_V$ and then $D^t(Z^1(M/_{\mathcal{F}})) \subset C_b^2(M/_{\mathcal{F}})$. Now let $\omega \in C_b^2(M/_{\mathcal{F}})$ which means that for an open subset V there is $f_V \in \Phi(V)$ such that $\omega\,|_V = j^{l,2}f_V$. A simple calculation proves that the basic 1-form locally given by $\theta\,|_V = df_V$ is globally defined on M and satisfies the relations $d\theta = 0$ and $\omega = D^t\theta$, hence $C_b^2(M/_{\mathcal{F}}) \subset D^t(Z^1(M/_{\mathcal{F}}))$.

So, the first equality is proved. An analogous argument is used for the second part of the relation (2.7).

The relation (2.7) shows that D^t induces an isomorphism between the quotient groups $Z^1(M/_{\mathcal{F}})/B^1(M/_{\mathcal{F}})$ and $C_b^2(M/_{\mathcal{F}})/E_b^2(M/_{\mathcal{F}})$. Hence the announced result is proved. ■

3 Fields of leafwise 2-jets

Let $\mathcal{F}$ be a foliation of dimension m and codimension n on the Riemannian manifold (M, g).

Definition 3.1. *Two maps $f, g \in \Omega^0(M)$ are said to determine the same (**homogeneous**) **leafwise 2- jet** at $x \in M$ if they define the same (homogeneous) 2-jet at x in the leaf $\mathcal{L}$ through x.*

Remark 3.1. (*a*) If $f, g \in \Omega^0(M)$ determine the same 2-jet at x, then they determine the same leafwise 2-jet at x. Indeed, if we consider $\mathcal{L}$ to be the leaf through x and i: $\mathcal{L} \to M$ the inclusion of submanifold $\mathcal{L}$ in M, for every curve $\sigma : R \to \mathcal{L}$ with $\sigma(0) = x$, it is easy to prove that the functions $f \circ i$ and $g \circ i$ determine the same 2-jet at x on $\mathcal{L}$.

(*b*) The reverse of (*a*) is not true. Indeed, for every basic function h, the functions f

and $f+h$ determine the same leafwise 2-jet at a point x, but they do not determine the same 2-jet on x.

Let f, g be two functions on M which determine the same leafwise 2-jet at x. From the definition, the maps f, g are zero at x and, for a distinguished chart U with local coordinates $(x^a, x^u)_{\substack{a=1,..,n \\ u=n+1,..,n+m}}$, in $U \cap \mathcal{L}$ they are satisfying:

$$\frac{\partial f}{\partial x^u}(x) = \frac{\partial g}{\partial x^u}(x); \quad \frac{\partial^2 f}{\partial x^u \partial x^v}(x) = \frac{\partial^2 g}{\partial x^u \partial x^v}(x)$$

The relation to "determine the same leafwise 2-jet at x" is an equivalence on $\Omega^0(M)$ and an equivalence class $j_x^{l,2} f$ is called the *leafwise 2-jet of f at x.*

We define the multiplication with a scalar α of a class and the product of two classes by $\alpha j_x^{l,2} f = j_x^{l,2}(\alpha f)$; $j_x^{l,2} f \cdot j_x^{l,2} g = j_x^{l,2}(f \cdot g)$.

We remark that $j_x^{l,2} x^a = 0$ and that the classes $j_x^{l,2} f$ and

$$\frac{\partial f}{\partial x^u}(x) j_x^{l,2} x^u + \frac{1}{2} \frac{\partial^2 f}{\partial x^u \partial x^v}(x) j_x^{l,2} x^u \cdot j_x^{l,2} x^v \tag{3.1}$$

are equal, hence (3.1) represent the local expression of $j_x^{l,2} f$.

For two distinguished charts with domains U and $\overline{U}$ with $U \cap \overline{U} \neq \emptyset$, from the Definition 3.1 we deduce the following relations

$$j_x^{l,2} \bar{x}^u = \frac{\partial \bar{x}^u}{\partial x^v}(x) j_x^{l,2} x^v + \frac{1}{2} \frac{\partial^2 \bar{x}^u}{\partial x^w \partial x^v}(x) j_x^{l,2} x^w \cdot j_x^{l,2} x^v$$

$$j_x^{l,2} \bar{x}^{u_1} \cdot j_x^{l,2} \bar{x}^{u_2} = \frac{\partial \bar{x}^{u_1}}{\partial x^{v_1}}(x) \frac{\partial \bar{x}^{u_2}}{\partial x^{v_2}}(x) j_x^{l,2} x^{v_1} \cdot j_x^{l,2} x^{v_2} \tag{3.2}$$

where $(\bar{x}^a, \bar{x}^u)$ are the local coordinates in the chart with domain $\overline{U}$. From the Remark 3.1 we have $j_x^2 f \subsetneqq j_x^{l,2} f$.

Generally, we say that $\omega_x^{l,2}$ is a *leafwise 2-jet at x on M* if it has the following expression in a distinguished chart:

$$\omega_x^{l,2} = \omega_u(x) \cdot j_x^{l,2} x^u + \frac{1}{2} \omega_{uv}(x) \cdot j_x^{l,2} x^u \cdot j_x^{l,2} x^v \tag{3.3}$$

where the coefficients ω_u, ω_{uv} are differentiable functions on M satisfying in $U \cap \bar{U} \cap \mathcal{L}$ $\omega_{uv} = \omega_{vu}$ and

$$\bar{\omega}_u = \omega_v \frac{\partial x^v}{\partial \bar{x}^u}; \qquad \bar{\omega}_{v_1 v_2} = \omega_{u_1 u_2} \frac{\partial x^{u_1}}{\partial \bar{x}^{v_1}} \frac{\partial x^{u_2}}{\partial \bar{x}^{v_2}} + \omega_u \frac{\partial^2 x^u}{\partial \bar{x}^{v_1} \partial \bar{x}^{v_2}} \tag{3.4}$$

We denote by $J_x^{l,2}(M)$ the space of leafwise 2-jets on M. $J^{l,2}(M) = \bigcup_{x \in M} J_x^{l,2}(M)$ is a fiber bundle over M with the fiber of dimension $\frac{m(m+3)}{2}$. The sections in this bundle are denoted by $J^2(\mathcal{F})$ and are called *fields of leafwise 2-jets* on M.

The map $j^{l,2} : \Omega^0(M) \to J^2(\mathcal{F})$ who assigns to a function f its field of leafwise 2-jets is called the *leafwise second differential* on M. This map is an algebras homomorphism.

From the relation (3.1) we obtain the local expression of the field of leafwise 2-jets of f:

$$j^{l,2}f = \frac{\partial f}{\partial x^u} j^{l,2}x^u + \frac{1}{2}\frac{\partial^2 f}{\partial x^u \partial x^v} j^{l,2}x^u \cdot j^{l,2}x^v \tag{3.5}$$

Remark 3.2. Let $(M', \mathcal{F}')$ be another Riemannian foliated manifold and $J^2(\mathcal{F}')$ the space of fields of leafwise 2-jets on M'. If $\mu : M \to M'$ is a foliated map, then we have the map $\mu_0 : \Omega^0(M') \to \Omega^0(M)$ defined by the relation $\mu_0(f) = f \circ \mu$.

For $x \in M$ let $U, (x^a, x^u)$ be a distinguished chart at x and $V, (y^b, y^v)$ be a distinguished chart at $\mu(x)$. We can associate to a field $j'^{l,2}f \in J^2(\mathcal{F}')$ the field $j^{l,2}(f \circ \mu) \in J^2(\mathcal{F})$; more generally, to a field $\omega'^2 \in J^2(\mathcal{F}')$ locally given by $\omega'^2 = \omega'_u \cdot j^{l,2}y^u + \frac{1}{2}\omega'_{uv} \cdot j^{l,2}y^u \cdot j^{l,2}y^v$, we associate the field $\mu^{l,2}(\omega'^2) \in J^2(\mathcal{F})$ locally given by

$$\mu^{l,2}(\omega'^2) = \omega'_u \cdot \frac{\partial y^u}{\partial x^{u_1}} j^{l,2}x^{u_1} + \frac{1}{2}\left(\omega'_{uv}\frac{\partial y^u}{\partial x^{u_1}}\frac{\partial y^v}{\partial x^{v_1}} + \omega'_u \frac{\partial^2 y^u}{\partial x^{u_1}\partial x^{v_1}}\right) \cdot j^{l,2}x^{u_1} \cdot j^{l,2}x^{v_1}$$

We obtained in this way a map $\mu^{l,2} : J^2(\mathcal{F}') \to J^2(\mathcal{F})$. If moreover μ is a diffeomorphism, then $\mu^{l,2}$ becomes an isomorphism.

Definition 3.2. *The field of leafwise 2-jets $\omega \in J^2(F)$ is called:*

(i) ***exact** if $\omega = j^{l,2}f$ for some $f \in \Omega^0(M)$;*

(ii) ***closed** if it is locally exact .*

We denote by $E^2(\mathcal{F})$ and $C^2(\mathcal{F})$ the spaces of exact, respectively closed fields of leafwise 2-jets on M and we have $E^2(\mathcal{F}) \subset C^2(\mathcal{F})$. We call the *cohomology group of fields of leafwise 2-jets on M* the following quotient group:

$$H^2_j(\mathcal{F}) = C^2(\mathcal{F})/E^2(\mathcal{F}) \tag{3.6}$$

Now we define a map between the space $\Omega^{0,1}(M)$ of $(0,1)$-forms on M and the space $J^2(\mathcal{F})$ of fields of leafwise 2-jets on M. Let (1.5) be an adapted cobasis. To every (0,1)-form ω locally given by $\omega = \omega_u \theta^u$, we associate the expression

$$D\omega = \omega_u \cdot j^{l,2}x^u + \frac{1}{4}\left(\frac{\partial \omega_u}{\partial x^v} + \frac{\partial \omega_v}{\partial x^u}\right) j^{l,2}x^u \cdot j^{l,2}x^v \tag{3.7}$$

Taking into account the transformation rule for the components of a $(0,1)$-form at the local chart changing, by a direct calculation we obtain that the coefficient functions of $D\omega$ are satisfying the relations (3.4). So (3.7) is a field of leafwise 2-jets. Moreover the map $D : \Omega^{0,1}(M) \longrightarrow J^2(\mathcal{F})$ defined by (3.7) is a monomorphism.

Theorem 3.1. *Let M be a Riemannian foliated manifold. The cohomology group of fields of leafwise 2-jets on M and the Čech cohomology group $H^1(M, \Phi)$ are isomorphic.*

Proof. We denote by $Z^{01}(M)$ and $B^{01}(M)$ the sets of d_{01}-closed and d_{01}-exact $(0,1)$-forms, respectively. We have the relations:

$$D(Z^{01}(M)) = C^2(\mathcal{F}), \quad D(B^{01}(M)) = E^2(\mathcal{F}) \tag{3.8}$$

Indeed, let be $\theta \in Z^{01}(M)$; from Poincaré lemma [7] for d_{01} we have that for an open subset V of M there is $f_V \in \Omega^0(V)$ such that $\theta\,|_V = d_{01} f_V = \frac{\partial f_V}{\partial x^u}\theta^u$. Taking into account the definition of the map D, we have $D\theta\,|_V = j^{l,2} f_V$ and then $D(Z^{01}(M)) \subset C^2(\mathcal{F})$. Now let $\omega \in C^2(\mathcal{F})$ which means that for an open subset V there is $f_V \in \Omega^0(V)$ such that $\omega\,|_V = j^{l,2} f_V$. A simple calculation proves that the $(0,1)$-form locally given by $\theta\,|_V = \frac{\partial f_V}{\partial x^u}\theta^u$ is globally defined on M and satisfies the relations $d_{01}\theta = 0$ and $\omega = D\theta$, hence $C^2(\mathcal{F}) \subset D(Z^{01}(M))$.

So, the first equality (3.8) is proved. An analogous argument is used for the second part of the relation (3.8). The relation (3.8) shows that D induces an isomorphism between the quotient groups $Z^{01}(M)/B^{01}(M)$ and $C^2(\mathcal{F})/E^2(\mathcal{F})$. From the definitions for $H^{01}(M)$ and $H^2_j(\mathcal{F})$ and from the Theorem 1.1, the announced result is proved. ■

References

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