LIBERTAS MATHEMATICA, vol XXIV (2004)

Geometrical Structures Associated to Second Order Dynamical Systems

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Abstract. In this paper we associate to implicit ordinary second order dynamical systems some geometrical structures (distinguished tensors and connections). We define a scalar product and we build two lifts on the tangent bundle of the fundamental distinguished tensor of the dynamical system.
Mathematics Subject Classification 2000 : 34C40, 53C07, 53C60.
Keywords: dynamical systems, nonlinear connections.

1. Let M be a m-dimensional manifold. We consider a system of second order differential equations, locally expressed by:

$$F_i(t,x,\dot{x},\ddot{x})=0,\quad i=\overline{1,m},\quad \text{with}\quad \det\left(\frac{\partial F_i}{\partial \ddot{x}^j}\right)\neq 0, \tag{1}$$

and a parameterized vector field: $X=\xi^i(t,x)\dfrac{\partial}{\partial x^i}$.

We also consider a transformation of local coordinates on M: $\bar{x}^i=\bar{x}^i(x^h)$ and the corresponding transformation on J^2M.

The functions F_i and the components of the vector $X(\xi^i)$ are changed by the rules: $\overline{F}_i=\dfrac{\partial x^h}{\partial \bar{x}^i}F_h$, respectively $\bar{\xi}^i=\dfrac{\partial \bar{x}^i}{\partial x^h}\xi^h$.

To the system (1) and to the vector field X we associate the system of variational forms:

$$M^X_{F,i}=\frac{\partial F_i}{\partial \ddot{x}^j}\frac{d^2\xi^j}{dt^2}+\frac{\partial F_i}{\partial \dot{x}^j}\frac{d\xi^j}{dt}+\frac{\partial F_i}{\partial x^j}\xi^j\overset{\text{not}}{=}a^0_{ij}\frac{d^2\xi^j}{dt^2}+a^1_{ij}\frac{d\xi^j}{dt}+a^2_{ij}\xi^j. \tag{2}$$

As is known [10], there is an unique adjoint system of forms:

$$\widetilde{M}^Y_{F,i}=\widetilde{a}^0_{ij}\frac{d^2\eta^j}{dt^2}+\widetilde{a}^1_{ij}\frac{d\eta^j}{dt}+\widetilde{a}^2_{ij}\eta^j, \tag{3}$$

where $Y=\eta^i(t,x)\dfrac{\partial}{\partial x^i}$ is another parameterized vector field, and the coefficients are given by:

$$\begin{cases}\widetilde{a}^0_{ij}=a^0_{ji},\\ \widetilde{a}^1_{ij}=2\dfrac{da^0_{ji}}{dt}-a^1_{ji},\\ \widetilde{a}^2_{ij}=\dfrac{d^2a^0_{ji}}{dt^2}-\dfrac{da^1_{ji}}{dt}+a^2_{ji}.\end{cases} \tag{4}$$

Proposition. *The coefficients a^0_{ij} are the components of a distinguished tensor of type (0,2) on M.*

The same property holds for the coefficients $\widetilde{a}^0_{ij}$ of the adjoint system.

Indeed, with respect to a coordinate transformation, the above coefficients are changed by the rules:

$$\bar{a}^0_{ij} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^j}a^0_{hk}, \text{ respectively } \bar{\widetilde{a}}^0_{ij} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^j}\widetilde{a}^0_{hk}.$$

We assume that $\det\left(\frac{\partial F_i}{\partial \ddot{x}^j}\right) \neq 0$. Then we can build the inverse matrix (a^{ij}_0) of $(a^0_{ij}) = \left(\frac{\partial F_i}{\partial \ddot{x}^j}\right)$. Analogously, for the adjoint system of forms (3), let $(\widetilde{a}^{ij}_0)$ be the inverse matrix of $(\widetilde{a}^0_{ij})$.

These elements are changed, under a change of a local chart, by the following rules:

$$\bar{a}^{ij}_0 = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\partial \bar{x}^j}{\partial x^k}a^{hk}_0, \text{ respectively } \bar{\widetilde{a}}^{ij}_0 = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\partial \bar{x}^j}{\partial x^k}\widetilde{a}^{hk}_0.$$

Let us define the coefficients M^i_j and $\widetilde{M}^i_j$ by the formulas:

$$\begin{cases} 2M^i_j = a^{ih}_0 a^1_{hj}, \\ 2\widetilde{M}^i_j = \widetilde{a}^{ih}_0 \widetilde{a}^1_{hj}. \end{cases} \tag{5}$$

Proposition. *The functions M^i_j are the components (parametrized) of a distinguished non-linear connection on the manifold M.*

The same property holds for the functions $\widetilde{M}^i_j$.

Indeed, these coefficients are changed, under a change of local chart, by:

$$\begin{cases} \dfrac{\partial x^p}{\partial \bar{x}^i}\overline{M}^i_j = M^p_h\dfrac{\partial x^h}{\partial \bar{x}^j} + \dfrac{\partial \dot{x}^p}{\partial \bar{x}^j}, \\ \dfrac{\partial x^p}{\partial \bar{x}^i}\overline{\widetilde{M}}^i_j = \widetilde{M}^p_h\dfrac{\partial x^h}{\partial \bar{x}^j} + \dfrac{\partial \dot{x}^p}{\partial \bar{x}^j}. \end{cases}$$

With these two sets of functions, we can build the functions:

$$T^i_j = \widetilde{M}^i_j - M^i_j, \tag{6}$$

which are changed, under a change of local coordinates, by:

$$\frac{\partial x^p}{\partial \bar{x}^i}\overline{T}^i_j = T^p_h\frac{\partial x^h}{\partial \bar{x}^j}.$$

It holds:

Proposition. *The functions T^i_j are the components of a distinguished tensor of type (1,1) on the manifold M.*

Remark. If the given system $F_i = 0$ is self-adjoint, we have $\widetilde{M}^i_j = M^i_j$. Then $T^i_j = 0$.

2. Let us consider two vector fields, $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$. The covariant derivative of the vector field X, with respect to the connection N determined by the coefficients M^i_j, is:

$$\frac{\delta X}{\delta t} = \frac{\delta X^i}{\delta t}\frac{\partial}{\partial x^i}, \text{ where } \frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + M^i_j X^j. \tag{7}$$

Under a change of local chart, we have:

$$\overline{\frac{\delta X^i}{\delta t}} = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\delta X^h}{\delta t}.$$

The vector field is parallel along a curve c, with respect to N, if its covariant derivative vanishes:

$$\frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + M^i_j X^j = 0. \tag{8}$$

By the definition of the coefficients M^i_j, (8) becomes:

$$\frac{dX^i}{dt} = -\frac{1}{2} a_0^{ih} a^1_{hj} X^j.$$

Similarly, the covariant derivative of the vector field Y, with respect to the connection $\widetilde{N}$ determined by the coefficients $\widetilde{M^i_j}$, is expressed by:

$$\frac{\widetilde{\delta Y^i}}{\delta t} = \frac{dY^i}{dt} + \widetilde{M^i_j} Y^j. \tag{9}$$

Using a^0_{ij} we can define a *scalar product*:

$$a^0_{ij} X^j Y^i. \tag{10}$$

Proposition. *If two vector fields* $X = X^i \frac{\partial}{\partial x^i}$ *and* $Y = Y^i \frac{\partial}{\partial x^i}$ *are parallel with respect to the connections* N *and respectively* $\widetilde{N}$, *we have:*

$$\frac{d}{dt}(a^0_{ij} X^j Y^i) = 0. \tag{11}$$

Indeed, the above relation becomes:

$$\frac{d}{dt}(a^0_{ij} X^j Y^i) = \left(\frac{da^0_{ij}}{dt} - \frac{1}{2} a^1_{ij} - \frac{1}{2}\widetilde{a}^1_{ji} \right) X^j Y^i,$$

which vanishes by the second adjunction relation from (4).

3. Proposition. *To the pair* a^0_{ij} *and* $\widetilde{a}^0_{ij}$ *of fundamental distinguished tensors on the manifold* M, *there corresponds two semi-distinguished tensors* A^C *and* A^D *(defined in what follows) on the manifold* TM, *called the complete and respectively the diagonal lift.*

We build the complete lift of the distinguished tensor $A = a^0_{ij} dx^i \otimes dx^j$:

$$A^C = \begin{pmatrix} \dfrac{da^0_{ij}}{dt} & a^0_{ij} \\ a^0_{ij} & 0 \end{pmatrix}. \tag{12}$$

By the relations (4) between the coefficients of the variational form M^X_F and those of the adjoint form $\widetilde{M}_F^Y$, we obtain for A^C the expression:

$$A^C = \begin{pmatrix} \frac{1}{2}(a_{ij}^1 + \widetilde{a}_{ji}^1) & \widetilde{a}_{ji}^0 \\ a_{ij}^0 & 0 \end{pmatrix}. \tag{13}$$

By (5), we can replace the coefficients a_{ij}^1 and $\widetilde{a}_{ji}^1$ in (13) and we get:

$$A^C = \begin{pmatrix} a_{ih}^0 M_j^h + \widetilde{a}_{jh}^0 \widetilde{M}_i^h & \widetilde{a}_{ji}^0 \\ a_{ij}^0 & 0 \end{pmatrix} = \begin{pmatrix} a_{ih}^0 M_j^h + a_{hj}^0 \widetilde{M}_i^h & a_{ij}^0 \\ a_{ij}^0 & 0 \end{pmatrix}. \tag{14}$$

With respect to a change of local coordinates, the following rule holds:

$$\begin{pmatrix} \bar{a}_{ih}^0 \overline{M}_j^h + \bar{\widetilde{a}}_{jh}^0 \overline{\widetilde{M}}_i^h & \bar{\widetilde{a}}_{ji}^0 \\ \bar{a}_{ij}^0 & 0 \end{pmatrix} = \tag{15}$$

$$= \begin{pmatrix} \frac{\partial x^p}{\partial \bar{x}^i} & 0 \\ \frac{\partial \dot{x}^p}{\partial \bar{x}^i} & \frac{\partial x^p}{\partial \bar{x}^i} \end{pmatrix}^t \begin{pmatrix} a_{pr}^0 M_q^r + \widetilde{a}_{qr}^0 \widetilde{M}_p^r & \widetilde{a}_{qp}^0 \\ a_{pq}^0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x^q}{\partial \bar{x}^j} & 0 \\ \frac{\partial \dot{x}^q}{\partial \bar{x}^j} & \frac{\partial \dot{x}^q}{\partial \bar{x}^j} \end{pmatrix}.$$

The matrix from (14) corresponds to the bilinear form:

$$a_{ij}^0 (dx^i \otimes \delta y^j + \widetilde{\delta} y^i \otimes dx^j), \tag{16}$$

where

$$\begin{cases} \delta y^h = dy^h + M_j^h dx^i, \\ \widetilde{\delta} y^h = dy^h + \widetilde{M}_i^h dx^i. \end{cases} \tag{17}$$

If we consider the bilinear form:

$$a_{ij}^0 (dx^i \otimes dx^j + \widetilde{\delta} y^i \otimes \delta y^j), \tag{18}$$

its corresponding matrix is successively written as:

$$\begin{pmatrix} a_{ij}^0 + a_{hk}^0 \widetilde{M}_i^h M_j^k & a_{hj}^0 \widetilde{M}_i^h \\ a_{ik}^0 M_j^k & a_{ij}^0 \end{pmatrix} = \begin{pmatrix} a_{ij}^0 + \frac{1}{4} a_0^{hk} \widetilde{a}_{hi}^1 a_{kj}^1 & \frac{1}{2} \widetilde{a}_{ji}^1 \\ \frac{1}{2} a_{ij}^1 & a_{ij}^0 \end{pmatrix}. \tag{19}$$

With respect to a change of local coordinates, we have:

$$\begin{pmatrix} \bar{a}_{ij}^0 + \bar{a}_{hk}^0 \overline{\widetilde{M}}_i^h \overline{M}_j^k & \bar{a}_{hj}^0 \overline{\widetilde{M}}_i^h \\ \bar{a}_{ik}^0 \overline{M}_j^k & \bar{a}_{ij}^0 \end{pmatrix} = \tag{20}$$

$$= \begin{pmatrix} \frac{\partial x^p}{\partial \bar{x}^i} & 0 \\ \frac{\partial \dot{x}^p}{\partial \bar{x}^i} & \frac{\partial x^p}{\partial \bar{x}^i} \end{pmatrix}^t \begin{pmatrix} a_{pq}^0 + a_{rs}^0 \widetilde{M}_p^r M_q^s & a_{rq}^0 \widetilde{M}_p^r \\ a_{ps}^0 M_q^s & a_{pq}^0 \end{pmatrix} \begin{pmatrix} \frac{\partial x^q}{\partial \bar{x}^j} & 0 \\ \frac{\partial \dot{x}^q}{\partial \bar{x}^j} & \frac{\partial x^q}{\partial \bar{x}^j} \end{pmatrix}.$$

These are two ways to define lifts of the distinguished tensor A. They extend the lifts known in the geometry of the osculator bundles. Both of them are closely related to the given system.

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