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Implicit First Order Dynamical Systems (I)

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Statement of the Problem

1. As is well-known [7], the necessary and sufficient condition for a first or second order dynamical system to be Lagrangian, respectively Hamiltonian (to admit the principle of minimal action) is that it has to be self-adjoint. A necessary condition for a second order system [1], respectively a first order system [3], to be self-adjoint, is that the system has to be written in the main form.

A dynamical system admits the variational principle if there is a Lagrange function such that the equations of the system are the Euler-Lagrange equations or equivalent with them. Generally, a second order system, written in the main form, is neither self-adjoint nor equivalent with a self-adjoint one, in the classical way, because there does not exist a Lagrange function for it $L : J^1M \to \mathbb{R}$. There always exist Lagrange functions $L : J^2M \to \mathbb{R}$ linear with respect to the accelerations, $L = L(t, x, \dot{x}, \ddot{x}) = A_i(t, x, \dot{x})\ddot{x}^i + B(t, x, \dot{x})$, such that the corresponding Euler-Lagrange equations (of the third order) are combinations of the equations of the given system and those of its derived system.

2. A first order system admits Lagrange functions to describe it: $L(t, x, \dot{x}) = A_i(t, x)\dot{x}^i + B(t, x)$, if and only if the number of its parameters is even. If the number of the state parameters of the system is odd, the method can be extended by adding an equation with a new state parameter [4]. So, the problem can be solved by the classical method.

3. If we consider a first order system (in the main or kinematical form) and its derived system, then the solutions of the first one are also solutions for the second one (they are among the solutions of the last one), as well as for any linear combination of them. Thus, in order to study a first order system we will consider a special second order system.

It was proved, locally in [5] and globally in [6], that Lagrange functions depending only on t, x and $\dot{x}$, quadratic in the components $\dot{x}^i$, can be found for such combinations as the above mentioned system, no matter the number of state parameters is.

For the proof of this result, it has been made the hypothesis that the configuration manifold has a Riemannian structure. This structure does not depend on the given system. The Lagrange function that results is depending, obviously, on the given system, but also on the Riemannian structure.

4. In what follows, we will show that to some implicit first order systems (which are not and cannot be made self-adjoint, or equivalent with self-adjoint ones) it is possible to associate a Lagrange function, with the property that the Euler-Lagrange equations, corresponding to it, are linear combinations of the equations of the given system and of its derived one. These equations are, obviously, of the second order written in main form. The Lagrange functions obtained in this way have special properties (they are conservation laws). In fact, the following theorem will be proved: Any holonomic dynamical system is Lagrangian.

1. Preliminaries

1.1. Notations

Let M be a smooth manifold of dimension m, defined by a coordinate chart $\mathcal{A} = \{(U,\varphi)\}$ and the bundle spaces TM, T^*M, J^nM ($J^0M = \mathbb{R}\times M$, $J^1M = \mathbb{R}\times TM$), $\delta^1_n = (E^1_n, p_\delta, J^nM)$, $(n\in\mathbb{N})$, where the last ones have, respectively, as bases the jet spaces J^nM and as total spaces $E^1_n = J^nM\times_M \Omega^1 M$, (where Ω^1M is the space of 1-forms); and endowed with the corresponding vectorial coordinate charts.

1.2. Systems of implicit n-order differential equations.

Let F^n be a section in δ^1_n:

$$F^n : (t,x,\dot{x},\ldots,x^{(n)}) \in J^nM \to F^n(t,x,\dot{x},\ldots,x^{(n)}) \in T^*_xM \tag{1.1}$$

with which we highlight the set:

$$\operatorname{Ker} F^n = \left\{(t,x,\dot{x},\ldots,x^{(n)}) \in J^nM \mid F^n(t,x,\dot{x},\ldots,x^{(n)}) = 0\right\}.$$

We will say that $\operatorname{Ker} F^n$ defines an *implicit n-order differential equation.*

In local coordinates (in a vectorial chart $(\overline{U},\Phi)$) the function F^n is given by:

$$F^n = F^n_i(t,x^h,\dot{x}^h,\ldots,x^{(n)h})dx^i, \quad \forall t\in I,\ (t,x,\dot{x},\ldots,x^{(n)}) \in J^nM. \tag{1.2}$$

The condition $F^n = 0$ is expressed by the relations:

$$F^n_i(t,x^h,\dot{x}^h,\ldots,x^{(n)h}) = 0, \quad (i,h=\overline{1,m}), \tag{1.3}$$

which is called *system of ordinary (non-degenerated) implicit n-order differential equations,* if they satisfy the condition: $\det\left(\dfrac{\partial F^n_i}{\partial x^{(n)j}}\right)_x \neq 0$, for any $x\in U$ (in what follows, we will write $\det\left(\dfrac{\partial F^n_i}{\partial x^{(n)j}}\right) \neq 0$), and *singular* if not.

The set $\operatorname{Ker} F^n$ (smooth submanifold in J^nM, of codimension m) leads us to the total space $\operatorname{Ker} F^n \times_M J^nM$ of the bundle space $\mathcal{F} = (\operatorname{Ker} F^n \times_M J^nM, \pi_n, J^nM)$, subbundle of $\delta^1_n = (J^nM\times_M T^*M, p_\tau, J^nM)$.

Under a change of coordinates on M: $\bar{x}^i = \bar{x}^i(x^h)$, the transformation rule for the local components $F^n_i(t,x^h,\dot{x}^h,\ldots,x^{(n)h})$ of F^n is:

$$\overline{F}^n_i = \frac{\partial x^h}{\partial \bar{x}^i}F^n_h, \tag{1.4}$$

hence they are the components of a distinct covector on the manifold M.

1.3. Solutions of n-order differential equations.

Let $c: t \in I \subset \mathbb{R} \to x = c(t) \in U \subset M$ be a differentiable curve on M. This is lifted to $J^n M$ by:

$$\bar{c}: t \in I \to \left(t, c(t), \frac{dc(t)}{dt}, \dots, \frac{d^n c(t)}{dt^n}\right) \in J^n M.$$

We will consider the reciprocal image of F^n by $\bar{c}$:

$$\bar{c}^* F^n = F^n \cdot \bar{c} = F^n \left[t, c(t), \frac{dc(t)}{dt}, \dots, \frac{d^n c(t)}{dt^n}\right].$$

Definition 1.1. *A function $c: t \in I \subset \mathbb{R} \to x = c(t) \in U \subset M$, which has the property:*

$$F_i^n \left[t, c(t), \frac{dc(t)}{dt}, \dots, \frac{d^n c(t)}{dt^n}\right] \equiv 0, \quad \forall t \in I, \ (i = \overline{1, m}),$$

is called ***local solution*** *for the differential system (1.3).*

In the case of ordinary differential equations, the implicit system (1.3) is equivalent (they have the same solutions) with the system:

$$x^{(n)i} = f^i(t, x, \dot{x}, \dots, x^{(n-1)}), \quad (i = \overline{1, m}), \tag{1.5}$$

which is written in kinematic form.

The system (1.5) defines a section f in the vectorial bundle $(J^n M, \pi^n_{n-1}, J^{n-1}M)$, where $\pi^n_{n-1}: (t, x, \dot{x}, \dots, x^{(n-1)}, x^{(n)}) \to (t, x, \dot{x}, \dots, x^{(n-1)})$.

2. Non-holonomic manifolds

2.1. Let ω be a Pfaff form, defined on $J^1 M$, locally written as:

$$\omega = F_i d\dot{x}^i + \Phi_i dx^i + f dt. \tag{2.1}$$

In general, the Pfaff equation $\omega = 0$ is not integrable, it does not admit as solutions submanifolds of maximal dimension $2m$ (hypermanifolds). But, it always admits as solutions, curves of the form $(x = c(t), \ \dot{x} = \gamma(t))$. The set of these curves (together with all submanifolds, solutions of $\omega = 0$) is called *non-holonomic manifold* (on $J^1 M$).

To the form ω (and to the equation $\omega = 0$) it is associated, in canonical way, the equation:

$$F_i \ddot{x}^i + \Phi_i \dot{x}^i + f = 0. \tag{2.2}$$

A curve on M: $x = c(t)$, $(t \in I)$, for which its lift $\bar{c}$ to $J^1 M$: $t \to \left(t, c(t), \frac{dc(t)}{dt}\right)$ is a solution for $\omega = 0$, is, obviously, a solution for (2.2):

$$F_i \left[t, c(t), \frac{dc(t)}{dt}\right] \frac{d^2 c^i(t)}{dt^2} + \Phi_i \left[t, c(t), \frac{dc(t)}{dt}\right] \frac{dc^i(t)}{dt} + f \left[t, c(t), \frac{dc(t)}{dt}\right] \equiv 0,$$

for any $t \in I$, and reciprocally.

2.2. We consider a transformation of local coordinates on M: $\bar{x}^i = \bar{x}^i(x^h)$ and the corresponding transformation on TM: $\bar{x}^i = \bar{x}^i(x^h)$, $\dot{\bar{x}}^i = \dfrac{\partial \bar{x}^i}{\partial x^h} \dot{x}^h$.

The local expressions of the form ω in two charts leads to:

$$\overline{F}_i d\dot{\overline{x}}^i + \overline{\Phi}_i d\overline{x}^i + \overline{f} dt = F_i d\dot{x}^i + \Phi_i dx^i + f dt.$$

Thus, we obtain the following transformation rules for the coefficients of the form ω:

$$\begin{cases} \overline{F}_i = \dfrac{\partial x^h}{\partial \overline{x}^i} F_h, \\ \overline{\Phi}_i = \dfrac{\partial \dot{x}^h}{\partial \overline{x}^i} F_h + \dfrac{\partial \dot{x}^h}{\partial \dot{\overline{x}}^i} \Phi_h, \\ \overline{f} = f. \end{cases} \tag{2.3}$$

By these transformation rules it results that the set of functions (F_i, Φ_i) are the components of a (non-autonomous) covector on TM.

The first formula of (2.3), allows us to say that the functions F_i are the components of an application:

$$\mathcal{F} : (t, x, \dot{x}) \in J^1 M \rightarrow \mathcal{F}(t, x, \dot{x}) := F_i(t, x, \dot{x}) dx^i \in T_x^* M,$$

(d-covector on M) and, as a result, its kernel $\operatorname{Ker} \mathcal{F}$ defines a submanifold in $J^1 M$ and, in the same time, a system of implicit differential equations:

$$F_i(t, x, \dot{x}) = 0. \tag{2.4}$$

The application $\omega \rightarrow F_i$ is not injective.

2.3. Let φ be a section in the vectorial bundle $J_0^1 = (J^1 M, \pi_0^1, J^0 M)$, where π_0^1 : $(t, x, \dot{x}) \rightarrow (t, x)$. The local expression of the section φ is given by the formula:

$$\dot{x}^i = \varphi^i(t, x) \tag{2.5}$$

and defines a first order system of ordinary differential equations, written in kinematic form.

3. Holonomic manifolds

3.1. We will consider a non-constant function $F : J^1 M \rightarrow \mathbb{R}$, which defines on $J^1 M$ a family of hypersurfaces, by the formula:

$$F(t, x, \dot{x}) = \lambda (const.), \tag{3.1}$$

family which is called *holonomic manifold* and we denote it by V. If, in particular, the form ω, considered in section **2**, is integrable ($\omega = dF$), then the manifold, solution for the equation $\omega = 0$, is holonomic and it is expressed by (3.1).

Let $c : t \in I \rightarrow x = c(t) \in M$ be a differentiable curve and $\bar{c} : t \in I \rightarrow \left(t, c(t), \dfrac{dc(t)}{dt}\right) \in J^1 M$ its lift to $J^1 M$. The curve $\bar{c}$ is on the holonomic manifold V if:

$$F\left[t, c(t), \frac{dc(t)}{dt}\right] \equiv \lambda, \quad \forall t \in I. \tag{3.2}$$

To the function F it is associated, in canonical way, the equation:

$$\frac{dF}{dt} = \frac{\partial F}{\partial \dot{x}^i} \ddot{x}^i + \frac{\partial F}{\partial x^i} \dot{x}^i + \frac{\partial F}{\partial t} = 0 \tag{3.3}$$

having the form (2.2), where:

$$F_i = \frac{\partial F}{\partial \dot{x}^i}, \ \Phi_i = \frac{\partial F}{\partial x^i} \text{ and } f = \frac{\partial F}{\partial t}. \tag{3.4}$$

A curve $x = c(t)$, with the property that its lift $\bar{c}$ is in V, is a solution for the equation (3.3) and reciprocally, any solution of the equation (3.3) is the lift of a curve $x = c(t)$.

3.2. We consider the restriction of the function F to any fiber of J_0^1, by:

$$F(t,x,\dot{x}) = F_{(t,x)}(\dot{x}) \tag{3.5}$$

and we suppose that in the critical points of $F_{(t,x)}$, those in which:

$$\frac{\partial F_{(t,x)}(\dot{x})}{\partial \dot{x}^i} = 0, \tag{3.6}$$

the Hessian of $F_{(t,x)}$ has the eigenvalues real positive numbers. These hypotheses are necessary and sufficient conditions for the solutions of the system (3.6) to define an optimal point for the function (3.5). We will say that such a function is optimal.

By using the notations (3.4): $F_i(t,x,\dot{x}) = \dfrac{\partial F_{(t,x)}(\dot{x})}{\partial \dot{x}^i} = \dfrac{\partial F(t,x,\dot{x})}{\partial \dot{x}^i}$, we obtain the implicit first order system of differential equations:

$$F_i(t,x,\dot{x}) = 0, \tag{3.7}$$

with the properties:

$$\frac{\partial F_i}{\partial \dot{x}^j} - \frac{\partial F_j}{\partial \dot{x}^i} = 0, \quad \det\left(\frac{\partial F_i}{\partial \dot{x}^j}\right) \neq 0. \tag{3.7$'$}$$

With the notations $A_{ij}(t,x,\dot{x}) = \dfrac{\partial F_i}{\partial \dot{x}^j}$, the properties (3.7′) can be rewritten as:

$$\det(A_{ij}) \neq 0, \quad A_{ij} = A_{ji}, \quad \frac{\partial A_{ih}}{\partial \dot{x}^j} = \frac{A_{jh}}{\partial \dot{x}^i}. \tag{3.7$''$}$$

Remark. If there is a function F which implies the existence of the system (3.7), then there is a whole family of such functions, which imply the same system.

3.3. By a transformation of local coordinates on M, the functions $F_i = \dfrac{\partial F}{\partial \dot{x}^i}$ change, according to (2.3_1), by the rule:

$$\frac{\partial \overline{F}}{\partial \dot{\bar{x}}^i} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\partial F}{\partial \dot{x}^h}. \tag{3.8}$$

Hence F_i are the components of a d-covector on M:

$$\widetilde{\alpha} = F_i dx^i = \frac{\partial F}{\partial \dot{x}^i} dx^i,$$

which defines a function (section in $(J^1M \times_M T^*M, p, J^1M)$):

$$\psi : J^1M \to J^1M \times_M T^*M,$$

whose kernel is expressed by the equations (3.7).

Theoretically, we can develop the system (3.7) in kinematic form (2.5), which has the property of defining a section φ.

Such an development cannot be always achieved, therefore there are to be studied first order systems written in the implicit form (3.7), for which the properties (3.4) and (3.6) hold.

3.4. Given a function $F : J^1M \to \mathbb{R}$, in the above mentioned conditions, we can associate to it an implicit first order system of differential equations (3.7) which verifies the relations $(3.7'_1)$. Reciprocally, we have:

Proposition 3.1. *If a system (3.7) verifies the conditions* $(3.7'_1)$, *then there is a function* F *such that:* $F_i = \dfrac{\partial F}{\partial \dot{x}^i}$.

Indeed, the system: $\dfrac{\partial F}{\partial \dot{x}^i} = F_i$ being completely integrable, it admits a solution. If $\alpha_{(t,x)}(\dot{x}) = \alpha = F_i d\dot{x}^i$ is a 1-form (parameterized by t and x), the condition for α to be closed is expressed by the relations $(3.7'_1)$ and, with respect to the Poincare's Lemma, we have the function:

$$F = \dot{x}^h \int_0^1 F_h(t, x, \tau\dot{x})d\tau. \tag{3.9}$$

The function F is one of the functions which imply the existence of the system (3.7).

3.5. Let us consider the function $F = F(t, x, \dot{x})$ as a Lagrange function, we associate to it the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i} = 0. \tag{3.10}$$

We consider the special case in which the solutions of the system (3.7) are solutions for the system (3.10). In this case they are also solutions for the system:

$$\Phi_i(t, x, \dot{x}) = \frac{\partial F}{\partial x^i} = 0. \tag{3.11}$$

From here, as a consequence, we have:

$$f = \frac{\partial F}{\partial t}_{/sol.} = 0. \tag{3.12}$$

By the integrability of the form $dF = F_i d\dot{x}^i + \Phi_i dx^i + f dt$, we have the relations:

(I) $$\frac{\partial F_i}{\partial \dot{x}^j} - \frac{\partial F_j}{\partial \dot{x}^i} = 0,$$

(II) $$\frac{\partial \Phi_i}{\partial \dot{x}^j} - \frac{\partial F_j}{\partial x^i} = 0,$$

(III) $$\frac{\partial \Phi_i}{\partial x^j} - \frac{\partial \Phi_j}{\partial x^i} = 0,$$

(IV) $$\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial F_i}{\partial t} = 0,$$

(V) $$\frac{\partial f}{\partial x^i} - \frac{\partial \Phi_i}{\partial t} = 0.$$

Definition 3.2. *An implicit dynamical system, given by the equations (3.7), is called* ***holonomic*** *if there is a holonomic manifold* V, *in* J^1M *(defined by a function* F *such that* $F_i = \dfrac{\partial F}{\partial \dot{x}^i}$*), which has the property that the lifts to* J^1M *of the solutions of the system,* *namely the curves* $t \in I \to \bar{c}(t) = \left(t, c(t), \dfrac{dc(t)}{dt}\right)$, *are on the fibers of the manifold* V, $(\forall t \in I)$.

Such a system satisfies the relations (I)-(V).

The function F, which defines the holonomic manifold V, is a *conservation law* for the given system (3.7).

In general, the system (3.7) is not self-adjoint.

Definition 3.3. *We will say that a system (3.7) is* ***semi-holonomic*** *if the functions* F_i *satisfy the relations (I).*

4. Self-adjoint systems

4.1. We consider the system (3.7) and we assume that it is not self-adjoint but satisfies the conditions (3.7′). We associate to it its derived system:

$$\frac{dF_i}{dt} = \frac{\partial F_i}{\partial \dot{x}^j}\ddot{x}^j + \frac{\partial F_i}{\partial x^j}\dot{x}^j + \frac{\partial F_i}{\partial t} = 0, \tag{4.1}$$

which is not, in general, self-adjoint. The system (4.1) is a system of second-order differential equations, written in main form, which has, as said before, the property that any solution of the system (3.7) is also solution for (4.1). Obviously, the same property holds for any linear combination of them, as:

$$\frac{dF_i}{dt} + C_i^j F_j = 0, \tag{4.2}$$

where $C = (C_i^j(t, x, \dot{x}))$ is an arbitrary matrix.

We look for the components C_i^j of the matrix C so that the system (4.2) to become a self-adjoint one. The system (4.2) is of the main form: $A_{ij}(t, x, \dot{x})\ddot{x}^j + B_i(t, x, \dot{x}) = 0$, where:

$$A_{ij} = \frac{\partial F_i}{\partial \dot{x}^j}, \quad B_i = \frac{\partial F_i}{\partial x^j}\dot{x}^j + \frac{\partial F_i}{\partial t} + C_i^j F_j. \tag{4.3}$$

Asking for (4.2) to be self-adjoint, the coefficients A_{ij} have to satisfy the properties (3.7″) and, by a change of coordinates, the transformation rule for them is:

$$\overline{A}_{ij} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \bar{x}^j}A_{hk}.$$

From here we can see that the functions A_{ij} are the components of a non-degenerated symmetric twice covariant d-tensor.

The second group of self-adjointness conditions for the system (4.2):

$$\frac{\partial B_i}{\partial \dot{x}^j} + \frac{\partial B_j}{\partial \dot{x}^i} = 2\left(\frac{\partial}{\partial t} + \dot{x}^h\frac{\partial}{\partial x^h}\right)A_{ij}, \tag{4.4}$$

leads us to the system:

$$\frac{\partial}{\partial \dot{x}^i}(C_j^h F_h) + \frac{\partial}{\partial \dot{x}^j}(C_i^h F_h) + \frac{\partial F_i}{\partial x^j} + \frac{\partial F_j}{\partial x^i} = 0.$$

By $(3.7_1')$, there exits a function F so that $F_i = \dfrac{\partial F}{\partial \dot{x}^i}$ and then the above system becomes:

$$\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial F}{\partial x^j} + C_j^h F_h\right) + \frac{\partial}{\partial \dot{x}^j}\left(\frac{\partial F}{\partial x^i} + C_i^h F_h\right) = 0. \tag{4.5}$$

A solution of this system is generated by the functions C_i^j satisfying the relations:

$$C_i^h F_h + \frac{\partial F}{\partial x^i} = 0, \quad \left(C_i^h \frac{\partial F}{\partial \dot{x}^h} + \frac{\partial F}{\partial x^i} = 0\right), \tag{4.6}$$

which is a linear system of m equations with m^2 unknown functions.

By (4.6), the last group of self-adjointness conditions:

$$\frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot{x}^h \frac{\partial}{\partial x^h}\right)\left(\frac{\partial B_i}{\partial \dot{x}^j} - \frac{\partial B_j}{\partial \dot{x}^i}\right) \tag{4.7}$$

is identically satisfied.

By (4.6), the equations (4.2) become now:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i} = 0, \tag{4.8}$$

which are the Euler-Lagrange equations corresponding to the Lagrangian F. We have:

Proposition 4.1. *Semi-holonomic implicit dynamical systems are Lagrangian (admit the variational principle).*

4.2. A more general solution to the equations (4.5) is:

$$C_i^h F_h + \frac{\partial F}{\partial x^i} + \varphi_i(t,x) = 0. \tag{4.6'}$$

The conditions (4.7) lead to the relations $\dfrac{\partial \varphi_i}{\partial x^j} - \dfrac{\partial \varphi_j}{\partial x^i} = 0$, from where, locally, $\varphi_i = \dfrac{\partial \varphi}{\partial x^i}$, where φ is an arbitrary function. Then, (4.2) becomes:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i} - \frac{\partial \varphi}{\partial x^i} = 0. \tag{4.7}$$

which, by the substitution $F_\varphi = F + \varphi$, leads to:

$$\frac{d}{dt}\left(\frac{\partial F_\varphi}{\partial \dot{x}^i}\right) - \frac{\partial F_\varphi}{\partial x^i} = 0. \tag{4.8}$$

Then we have a family of Lagrange functions F_φ.

4.3. We consider now the case of a system (3.7), for which the conditions $(3.7_1')$ are not verified, and, together with it, the set of the equivalent systems:

$$G_i = D_i^j F_j = 0, \quad \det(D_i^j) \neq 0. \tag{4.9}$$

It holds:

Proposition 4.2. *If there is a conservation law G for the system (3.7), then, in the set of equivalent systems of implicit first order equations (4.9) to (3.7), there is a system $G_i = 0$ (with $G_i = \dfrac{\partial G}{\partial \dot{x}^i}$) for which the property: $\dfrac{\partial G_i}{\partial \dot{x}^j} - \dfrac{\partial G_j}{\partial \dot{x}^i} = 0$ holds.*

Proof. We consider the system (3.7) and we assume that the conditions $(3.7_1')$ are not verified. Let $G = \lambda$ be a non-constant conservation law for the system (3.7) (G is a function on J^1M, which has the property that $\frac{dG}{dt}_{/sol.} = 0$.

We denote by $G_i = \frac{\partial G}{\partial \dot{x}^i}$ and we look for a matrix (D_i^j) so that the functions D_i^j to verify $D_i^j F_j = G_i$. For any index i fixed, from the m functions D_i^j $(j = \overline{1,m})$ we can choose $m-1$ of them arbitrary: $D_i^1, \dots, \widehat{D}_i^p, \dots, D_i^m$ and the last one will be given by the relation:

$$D_i^p = \frac{1}{F_p}\left[G_i - (D_i^1 F_1 + \cdots + \widehat{D_i^p F_p} + \cdots + D_i^m F_m)\right].$$

The system of functions D_i^j, established in this way, satisfies the request conditions $(3.7_1')$. If $\det(D_i^j) = 0$, we choose two such solutions $\underset{1}{D}{}_i^j$, $\underset{2}{D}{}_i^j$ and we consider the linear combination $\lambda^1 \underset{1}{D}{}_i^j + \lambda^2 \underset{2}{D}{}_i^j$ which has the property that: $\left(\lambda^1 \underset{1}{D}{}_i^j + \lambda^2 \underset{2}{D}{}_i^j\right) F_j = (\lambda^1 + \lambda^2) G_i$. Obviously, these combinations verify the conditions $(3.7_1')$ for any λ^1 and λ^2 such that $\det\left(\lambda^1 \underset{1}{D}{}_i^j + \lambda^2 \underset{2}{D}{}_i^j\right) \neq 0$.

4.4. As already specified in **4.1**, in general, a dynamical system (3.7) is Lagrangian if there is a self-adjoint linear combination:

$$D_i^j \frac{dF_j}{dt} + E_i^j F_j = 0. \tag{4.10}$$

Proposition 4.3. *The condition that the system (4.10) to be self-adjoint is equivalent with the existence of a matrix (D_i^j), with $\det(D_i^j) \neq 0$, so that the equivalent system $G_i = D_i^j F_j = 0$ to the system $F_i = 0$, satisfies the relation:*

$$\frac{\partial G_i}{\partial \dot{x}^j} - \frac{\partial G_j}{\partial \dot{x}^i} = 0.$$

Indeed, the system (4.10) can be written as:

$$\frac{d(D_i^j F_j)}{dt} + \left(E_i^h - \frac{dD_i^h}{dt}\right)\overline{D}_h^j (D_j^k F_k) = 0, \tag{4.11}$$

which has the form (4.2), in the functions G_i.

The self-adjointness conditions for (4.11) ask that:

$$\frac{\partial (D_i^p F_p)}{\partial \dot{x}^j} - \frac{\partial (D_j^p F_p)}{\partial \dot{x}^i} = 0.$$

These relations, written in full form as:

$$\left(\frac{\partial D_i^p}{\partial \dot{x}^j} - \frac{\partial D_j^p}{\partial \dot{x}^i}\right) F_p + D_i^p \frac{\partial F_p}{\partial \dot{x}^j} - D_j^p \frac{\partial F_p}{\partial \dot{x}^i} = 0, \tag{4.12}$$

form a system of $\frac{m(m-1)}{2}$ equations with m^2 unknown functions D_i^j, for which a solution $D_i^j = 0$ with $\det(D_i^j) \neq 0$ was given in **4.3**.

Proposition 4.4. *A necessary and sufficient condition that the system (3.7) to be equivalent with a semi-holonomic system is that the system to admit a solution (D_i^j) with $\det(D_i^j) \neq 0$.*

5. Lagrange functions

5.1. By the above judgments, it follows that any semi-holonomic first order dynamical system (3.7), with meaning of (3.4_1), satisfies the properties $(3.7'_1)$. We will prove that the properties $(3.7'_1)$ are characteristic.

Proposition 5.1. *Any implicit first order dynamical system (3.7):*

$$F_i(t, x, \dot{x}) = 0, \tag{5.1}$$

satisfying the conditions $(3.7')$*:*

$$\frac{\partial F_i}{\partial \dot{x}^j} - \frac{\partial F_j}{\partial \dot{x}^i} = 0, \quad \det\left(\frac{\partial F_i}{\partial \dot{x}^j}\right) \neq 0, \tag{5.2}$$

is locally Lagrangian.

We will prove that there exists a function $L(t, x, \dot{x})$ such that on the holonomic manifold V, defined by the equation $L = \lambda$, there are all the solutions of the given system (3.7), lifted on J^1M, the function L being a Lagrangian of the system (4.9).

To the system (5.1) we associate a Pfaff form (2.1):

$$\omega = F_i d\dot{x}^i + \Phi_i dx^i + f\,dt, \tag{5.3}$$

where the functions F_i are the left hand sides of the equations (5.1), satisfying the relations (5.2), and the functions Φ_i and f are, for moment, arbitrary.

We look for the coefficients Φ_i and f so that the form ω to be closed ($d\omega = 0$). This condition is the same as the relations (I)-(V) from **3.5**.

The following properties hold:

1. The relations (I) are satisfied by the assumptions (5.2).

2. The system of m^2 equations with partial derivative (II) can be decomposed in m systems (if the index i is fixed), each of them being completely integrable.

Indeed, by (I), we have:

$$\frac{\partial^2 \Phi_i}{\partial \dot{x}^j \partial \dot{x}^h} - \frac{\partial^2 \Phi_i}{\partial \dot{x}^h \partial \dot{x}^j} = \frac{\partial}{\partial x^i}\left(\frac{\partial F_h}{\partial \dot{x}^j} - \frac{\partial F_j}{\partial \dot{x}^h}\right) \overset{\mathrm{I}}{=} 0.$$

3. We will also see that the relations (III) are satisfied.

4. The equations (IV) and (V) form a completely integrable system. Indeed,

$$\frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} - \frac{\partial^2 f}{\partial \dot{x}^j \partial \dot{x}^i} = \frac{\partial}{\partial t}\left(\frac{\partial F_j}{\partial \dot{x}^i} - \frac{\partial F_i}{\partial \dot{x}^j}\right) \overset{\mathrm{I}}{=} 0,$$

$$\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial}{\partial t}\left(\frac{\partial \Phi_j}{\partial x^i} - \frac{\partial \Phi_i}{\partial x^j}\right) \overset{\mathrm{III}}{=} 0,$$

$$\frac{\partial^2 f}{\partial \dot{x}^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial \dot{x}^i} = \frac{\partial}{\partial t}\left(\frac{\partial \Phi_j}{\partial \dot{x}^i} - \frac{\partial F_i}{\partial x^j}\right) \overset{\mathrm{II}}{=} 0.$$

Consequently, the form ω, (where the functions F_i are given by (5.1) and the functions Φ_i and f are solutions of the system (II)-(III), respectively (IV)-(V)), being closed, it is locally exact (exact on any contractible domain). Hence there is a function F, with $\omega = dF$, satisfying all the previous conditions. This proves the proposition.

5.2. In order to integrate the system (II)-(V), we will make the following statements:

The system (II), with i fixed, being integrable, there are closed 1-forms:

$$\alpha_i = \frac{\partial F_h}{\partial x^i} d\dot{x}^h = d_{(t,x)}\Phi_i, \tag{5.4}$$

where t and x are considered as parameters.

By Poincare's Lemma, the functions Φ_i (for any i) are given by:

$$\Phi_i = \dot{x}^h \int_0^1 \frac{\partial F_h}{\partial x^i}(t, x, \tau\dot{x}) d\tau. \tag{5.5}$$

These functions, solutions of the system (II), verify the relations (III), hence the claim in **5.2**.

A similar statement for the completely integrable system (IV)-(V), leads us to a closed 1-form (parameterized by t):

$$\alpha = \frac{\partial F_i}{\partial t} d\dot{x}^i + \frac{\partial \Phi_i}{\partial t} dx^i = d_{(t)} f. \tag{5.6}$$

By Poincare's Lemma, we have:

$$f = \dot{x}^h \int_0^1 \frac{\partial F_h}{\partial t}(t, \tau x, \tau\dot{x}) d\tau + x^h \int_0^1 \frac{\partial \Phi_h}{\partial t}(t, \tau x, \tau\dot{x}) d\tau. \tag{5.7}$$

By (5.5) and (5.7), the form ω, given by (5.3), is closed and hence locally it is the derivative of a function L defined, by Poincare's Lemma, as:

$$L = \dot{x}^h \int_0^1 \frac{\partial F_h}{\partial t}(\tau t, \tau x, \tau\dot{x}) d\tau + x^h \int_0^1 \frac{\partial \Phi_h}{\partial t}(\tau t, \tau x, \tau\dot{x}) d\tau + t \int_0^1 f(\tau t, \tau x, \tau\dot{x}) d\tau. \tag{5.8}$$

The relation (5.8) let us to construct effectively the Lagrange function L.

The Lagrange function L, obtained in this way, is a conservation law for the system (3.7).

Theorem 5.2. *The necessary and sufficient condition for an implicit first order dynamical system (5.1) to be holonomic-Lagrangian (to admit a function $L(t, x, \dot{x})$ which is at the same time conservation law and Lagrange function), is that the relations (5.2) to be satisfied.*

6. Lagrange spaces

Let $L : TM \to \mathbb{R}$ be a non-degenerated autonomous Lagrange function ($\det\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\right) \neq 0$). The functions $A_{ij}(x, \dot{x}) = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ are the covariant components of a two times covariant and non-degenerated d-tensor, satisfying the properties (3.7″):

$$\det(A_{ij}) \neq 0, \quad A_{ij} = A_{ji}, \quad \frac{\partial A_{ih}}{\partial \dot{x}^j} = \frac{A_{jh}}{\partial \dot{x}^i}. \tag{6.1}$$

We say that the function L defines on the manifold M a *Lagrange space structure.*

We call *generalized Lagrange space*, a set (M, A), where $A = (A_{ij})$ is a twice covariant, symmetric and non-degenerated d-tensor (it satisfies the first and second conditions from (6.1)).

The tensor A_{ij} is called *fundamental tensor* of the space.

Remark. By the above definitions, it follows that any Lagrange space is an example of generalized Lagrange space. It holds:

Theorem 6.1. *A generalized Lagrange space is a Lagrange space if and only if its fundamental tensor satisfies the relations* (6.1_3).

Proof. By the remark, the necessity of the condition is immediate. We have to prove that any generalized Lagrange space is a Lagrange space if its fundamental tensor satisfies the relations (6.1_3).

If in the system:

$$\frac{\partial F_i}{\partial \dot{x}^j} = A_{ij}, \tag{6.2}$$

the index i is fixed, an integrable system follows (by the condition (6.1_3)), which admits the solution:

$$F_i = \dot{x}^h \int_0^1 A_{ih}(t, x, \tau\dot{x}) d\tau. \tag{6.3}$$

These functions F_i, satisfy the relations (5.2). Hence according to **3**, they define a holonomic-Lagrangian manifold and therefore a Lagrange function L which has the property $\dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = A_{ij}(t, x, \dot{x})$.

Interpretation. We may view the functions F_i, obtained from L by $F_i = \dfrac{\partial L}{\partial \dot{x}^i}$, as generalized impulses. The condition $F_i = 0$ defines the evolution of a dynamical system.

In the geometrical structure associated to the system, its evolution can be considered as being "inertial" because the impulse vanishes.

The function $-\varphi = V$ is viewed as potential energy and L can be considered as kinetic energy.

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