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Curvature Tensors on Complex Lagrange Spaces

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Abstract. In this paper, our aim is to give an expression on $T_C(T'M)$ of the Levi-Civita connection for the Hermitian metric in a complex Lagrange space and to investigate its curvature tensor.

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1 Introduction

The study of the complex Lagrange geometry was recently developed by Gh. Munteanu ([6, 7, 8]). The holomorphic sectional curvature in a complex Finsler space was actually studied only on the horizontal part of Riemannian curvature ([1, 2]). In this paper, we begin a study of curvature tensors using the Sasaki lift on $T_C(T'M)$ of a general Lagrange metric tensor.

Let M be a complex manifold, $\dim_C M = n$, and let (U, z^k) be the complex coordinates in a local chart. The complexification $T_C M$ of the tangent bundle TM is decomposed in each $z \in M$ as $T_C M = T'M \oplus T''M$, where $T'M$ is the holomorphic bundle in which, as a complex manifold, a point is $u = (z^k, \eta^k)$ in a local chart. Let $\pi : T'M \to M$ be the canonical projection, $V(T'M) = \{\xi \in T(T'M)/\pi_*(\xi) = 0\}$ be the vertical subbundle, and $\mathcal{V}(T'M)$ the module of its vertical sections ([1, 2, 6]), spanned by $\{\dot{\partial} =: \frac{\dot{\partial}}{\partial \eta^k}\}$.

A complex nonlinear connection, $(c.n.c.)$, is a supplementary subbundle to $V(T'M)$ in $T'(T'M)$, i.e.,

$$T'(T'M) = H(T'M) \oplus V(T'M) \tag{1.1}$$

and by $\mathcal{H}(T'M)$ we denote the horizontal distribution, in which an adapted base of the $(c.n.c.)$ is $\{\delta_k =: \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$.

By conjugation we obtain a decomposition of whole $T_C(T'M)$:

$$T_C(T'M) = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)} \tag{1.2}$$

and the corresponding conjugate bases will be denoted by $\{\delta_{\bar{k}} =: \frac{\delta}{\delta \bar{z}^k}\}, \{\dot{\partial}_{\bar{k}} =: \frac{\dot{\partial}}{\partial \bar{\eta}^k}\}$.

2 The Levi-Civita connection on $T'M$

Let (M,L) be a complex Lagrange space, i.e., $L : T'M \to R$ is a smooth function for which the metric tensor $g_{i\bar{j}} = \dot{\partial}^2 L/\dot{\partial}\eta^i \dot{\partial}\bar{\eta}^j$ is nondegenerate. We shall use through the Chern-Lagrange *c.n.c.*, introduced and studied by Gh. Munteanu ([8]):

$$\overset{CL}{N_j^i} = g^{\bar{m}i} \frac{\dot{\partial}^2 L}{\dot{\partial} z^j \dot{\partial}\bar{\eta}^m} \tag{2.1}$$

in respect to which the corresponding adapted frames satisfy the following interesting relations

$$[\delta_j, \delta_k] = 0\ ,\ \delta_j(\overset{CL}{N_k^i}) = \delta_k(\overset{CL}{N_j^i})\ ,\ (\dot{\partial}_{\bar{l}} \overset{CL}{N_j^i}) g_{i\bar{k}} - (\dot{\partial}_{\bar{k}} \overset{CL}{N_j^i}) g_{i\bar{l}} = 0$$

$$(\dot{\partial}_{\bar{l}} \overset{CL}{N_j^i}) g_{i\bar{k}} + (\dot{\partial}_{\bar{k}} \overset{CL}{N_j^i}) g_{i\bar{l}} = 2(\dot{\partial}_{\bar{k}} g_{i\bar{l}}) \overset{CL}{N_j^i}\ ,\ (\dot{\partial}_l \overset{CL}{N_j^i}) g_{i\bar{k}} = \delta_j g_{l\bar{k}} \tag{2.2}$$

In particular, when L is positively and absolutely homogeneous of degree two, $L(z, \lambda\eta) = |\lambda|^2 L(z,\eta)$, $\lambda \in \mathbb{C}$, then (M,L) is a complex Finsler space and $\overset{CL}{N_j^i} = g^{\bar{m}i} \frac{\dot{\partial} g_{k\bar{m}}}{\partial z^j} \eta^k$ is just the Chern-Finsler connection ([1, 2]).

We consider now the Hermitian metrical structure $\mathbf{G}$ on $T_C(T'M)$, ([6]):

$$\mathbf{G} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j \tag{2.3}$$

where $\{dz^k, \delta\eta^k\}$ is the adapted cobase of *c.n.c.*.

First of all, we compute the coefficients of the Levi-Civita connection ∇ of in adapted frame $\{\delta_k, \dot{\partial}_k, \delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$, following similar path as Anastasiei-Shimada did in the real case ([3]).

$$\begin{aligned}
\nabla_{\delta_k} \delta_j &= \overset{1}{L}{}_{jk}^i \delta_i + \overset{2}{A}{}_{jk}^i \dot{\partial}_i + \overset{3}{A}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{A}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\delta_k} \dot{\partial}_j &= \overset{1}{B}{}_{jk}^i \delta_i + \overset{2}{L}{}_{jk}^i \dot{\partial}_i + \overset{3}{B}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{B}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\delta_k} \delta_{\bar{j}} &= \overset{1}{D}{}_{jk}^i \delta_i + \overset{2}{D}{}_{jk}^i \dot{\partial}_i + \overset{3}{L}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{D}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\delta_k} \dot{\partial}_{\bar{j}} &= \overset{1}{E}{}_{jk}^i \delta_i + \overset{2}{E}{}_{jk}^i \dot{\partial}_i + \overset{3}{E}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{L}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\dot{\partial}_k} \delta_j &= \overset{1}{C}{}_{jk}^i \delta_i + \overset{2}{F}{}_{jk}^i \dot{\partial}_i + \overset{3}{F}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{F}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\dot{\partial}_k} \dot{\partial}_j &= \overset{1}{G}{}_{jk}^i \delta_i + \overset{2}{C}{}_{jk}^i \dot{\partial}_i + \overset{3}{G}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{G}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\dot{\partial}_k} \delta_{\bar{j}} &= \overset{1}{H}{}_{jk}^i \delta_i + \overset{2}{H}{}_{jk}^i \dot{\partial}_i + \overset{3}{C}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{H}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}} \\
\nabla_{\dot{\partial}_k} \dot{\partial}_{\bar{j}} &= \overset{1}{M}{}_{jk}^i \delta_i + \overset{2}{M}{}_{jk}^i \dot{\partial}_i + \overset{3}{M}{}_{jk}^{\bar{i}} \delta_{\bar{i}} + \overset{4}{C}{}_{jk}^{\bar{i}} \dot{\partial}_{\bar{i}}
\end{aligned} \tag{2.4}$$

and their conjugates, since $\nabla_{\overline{X}} \overline{Y} = \overline{\nabla_X Y}$.

Let $\mathbb{T}$ be the torsion of ∇, i.e. $\mathbb{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, where X, Y are vector fields on $T_C(T'M)$. The condition ∇ is torsion-free is equivalent to

$$\mathbb{T}(\delta_j, \delta_k) = \mathbb{T}(\delta_j, \delta_{\bar{k}}) = \mathbb{T}(\delta_j, \dot{\partial}_k) = \mathbb{T}(\delta_{\bar{j}}, \dot{\partial}_k) = \mathbb{T}(\dot{\partial}_j, \dot{\partial}_k) = \mathbb{T}(\dot{\partial}_j, \dot{\partial}_{\bar{k}}) = 0 \tag{2.5}$$

and their conjugates. Using the following formulas

$$[\delta_j,\delta_k]=R^i_{jk}\dot{\partial}_i,\ [\delta_j,\delta_{\bar{k}}]=(\delta_{\bar{k}}N^i_j)\dot{\partial}_i-(\delta_j\overline{N}^{\bar{i}}_{\bar{k}})\dot{\partial}_{\bar{i}},\ [\delta_j,\dot{\partial}_k]=(\dot{\partial}_kN^i_j)\dot{\partial}_i$$
$$[\delta_{\bar{j}},\dot{\partial}_k]=(\dot{\partial}_k\bar{N}^{\bar{i}}_{\bar{j}})\dot{\partial}_{\bar{i}},\ [\dot{\partial}_j,\dot{\partial}_k]=0,\ [\dot{\partial}_j,\dot{\partial}_{\bar{k}}]=0, \tag{2.6}$$

where $R^i_{jk}=\delta_kN^i_j-\delta_jN^i_k=0$ for the Chern-Lagrange *c.n.c.*, and their conjugates, by direct computation, requiring (2.5) and $\nabla\mathbf{G}=0$, we obtain

Theorem 2.1. *The local coefficients of the Levi-Civita connection of the Hermitian metrical structure* $\mathbf{G}$ *in the frame* $\{\delta_i,\dot{\partial}_i,\delta_{\bar{i}},\dot{\partial}_{\bar{i}}\}$ *are as follows:*

$$\overset{1}{L}{}^i_{jk}=\frac{1}{2}g^{\bar{l}i}(\delta_kg_{j\bar{l}}+\delta_jg_{k\bar{l}})\ ,\ \overset{2}{A}{}^i_{jk}=\overset{3}{A}{}^{\bar{i}}_{jk}=\overset{4}{A}{}^{\bar{i}}_{jk}=0 \tag{2.7}$$
$$\overset{1}{B}{}^i_{jk}=\frac{1}{2}g^{\bar{l}i}(g_{j\bar{h}}\delta_k\overset{\overline{CL}}{N}{}^{\bar{h}}_{\bar{l}}+\dot{\partial}_jg_{k\bar{l}})=\overset{1}{C}{}^i_{kj}\ ,\ \overset{2}{L}{}^i_{jk}=g^{\bar{l}i}\delta_kg_{j\bar{l}},\ \overset{3}{B}{}^{\bar{i}}_{jk}=\overset{4}{B}{}^{\bar{i}}_{jk}=0,$$
$$\overset{1}{D}{}^i_{\bar{j}k}=\frac{1}{2}g^{\bar{l}i}(\delta_{\bar{j}}g_{j\bar{l}}+\delta_{\bar{l}}g_{k\bar{j}})=\overset{3}{L}{}^i_{k\bar{j}}\ ,\ \ \overset{2}{D}{}^i_{\bar{j}k}=\frac{1}{2}g^{\bar{l}i}(g_{h\bar{l}}\delta_{\bar{j}}\overset{\overline{CL}}{N}{}^h_k-\dot{\partial}_{\bar{l}}g_{k\bar{j}})$$
$$\overset{4}{D}{}^{\bar{i}}_{jk}=\frac{1}{2}g^{\bar{i}l}(g_{l\bar{h}}\delta_k\overset{\overline{CL}}{N}{}^{\bar{h}}_{\bar{j}}+\dot{\partial}_lg_{k\bar{j}})\ \ ,\ \ \overset{2}{F}{}^i_{jk}=\overset{3}{F}{}^{\bar{i}}_{jk}=\overset{4}{F}{}^{\bar{i}}_{jk}=0$$
$$\overset{1}{E}{}^i_{\bar{j}k}=\frac{-1}{2}g^{\bar{l}i}(g_{h\bar{j}}\delta_{\bar{l}}\overset{CL}{N}{}^h_k-\dot{\partial}_jg_{k\bar{l}})=\overset{3}{C}{}^i_{k\bar{j}}\ ,\overset{2}{E}{}^i_{jk}=\overset{3}{E}{}^{\bar{i}}_{jk}=\overset{4}{L}{}^{\bar{i}}_{jk}=0$$
$$\overset{1}{G}{}^i_{jk}=g^{\bar{l}i}\dot{\partial}_kg_{j\bar{h}}\overset{\overline{CL}}{N}{}^{\bar{h}}_{\bar{l}}\ ,\ \overset{2}{C}{}^i_{jk}=g^{\bar{l}i}\dot{\partial}_kg_{j\bar{l}}\ ,\ \overset{3}{G}{}^{\bar{i}}_{jk}=\overset{4}{G}{}^{\bar{i}}_{jk}=0$$
$$\overset{4}{H}{}^{\bar{i}}_{jk}=-g^{\bar{i}l}\dot{\partial}_kg_{l\bar{h}}\overset{\overline{CL}}{N}{}^{\bar{h}}_{\bar{j}}\ ,\ \overset{1}{H}{}^i_{jk}=\overset{2}{H}{}^i_{jk}=\overset{1}{M}{}^i_{jk}=\overset{2}{M}{}^i_{jk}=\overset{4}{M}{}^{\bar{i}}_{jk}=\overset{4}{C}{}^i_{jk}=0$$

and the conjugates.

Remark 2.1. In particular, if $\delta_kg_{j\bar{l}}=\delta_jg_{k\bar{l}}$, i.e. the metric tensor is strongly Kähler, ([1]), then $\overset{1}{L}{}^i_{jk}=\overset{2}{L}{}^i_{jk}$, and $\overset{1}{D}{}^i_{\bar{j}k}=\overset{3}{L}{}^i_{j\bar{k}}=0$.

3 The curvature tensor of the Levi-Civita connection on $T'M$

As in real case, ([3]), we shall compute the components of the curvatures of ∇ using an intermediate arbitrary $d-$connection, D on $T_C(T'M)$ ([6]):

$$D_{\delta_k}\delta_j=\overset{1}{L}{}^i_{jk}\delta_i\ ,\ D_{\delta_k}\dot{\partial}_j=\overset{2}{L}{}^i_{jk}\dot{\partial}_i\ ,\ D_{\delta_k}\delta_{\bar{j}}=\overset{3}{L}{}^{\bar{i}}_{\bar{j}k}\delta_{\bar{i}}\ ,\ D_{\delta_k}\dot{\partial}_{\bar{j}}=\overset{4}{L}{}^{\bar{i}}_{\bar{j}k}\dot{\partial}_{\bar{i}} \tag{3.1}$$
$$D_{\dot{\partial}_k}\delta_j=\overset{1}{C}{}^i_{jk}\delta_i\ ,\ D_{\dot{\partial}_k}\dot{\partial}_j=\overset{2}{C}{}^i_{jk}\dot{\partial}_i\ ,\ D_{\dot{\partial}_k}\delta_{\bar{j}}=\overset{3}{C}{}^{\bar{i}}_{\bar{j}k}\delta_{\bar{i}}\ ,\ D_{\dot{\partial}_k}\dot{\partial}_{\bar{j}}=\overset{4}{C}{}^{\bar{i}}_{\bar{j}k}\dot{\partial}_{\bar{i}}$$

and their conjugates.

Let us express the curvature components of D, which is not free torsion, in adapted base.

$$
\begin{aligned}
&R(\delta_h,\delta_k)\delta_j=\widetilde{R}^{i}_{jkh}\delta_i\ ,\ R(\delta_h,\delta_k)\delta_{\bar j}=\widetilde{R}^{\bar i}_{\bar jkh}\delta_{\bar i}\ ,\ R(\delta_h,\delta_{\bar k})\delta_{\bar j}=\widetilde{R}^{\bar i}_{\bar j\bar kh}\delta_{\bar i}\\
&R(\delta_h,\delta_k)\dot\partial_j=\widetilde{\Omega}^{i}_{jkh}\dot\partial_i\ ,\ R(\delta_h,\delta_k)\dot\partial_{\bar j}=\widetilde{\Omega}^{\bar i}_{\bar jkh}\dot\partial_{\bar i}\ ,\ R(\delta_h,\delta_{\bar k})\dot\partial_{\bar j}=\widetilde{\Omega}^{\bar i}_{\bar j\bar kh}\dot\partial_{\bar i}\\
&R(\delta_h,\dot\partial_k)\delta_j=\widetilde{\Pi}^{i}_{jkh}\delta_i\ ,\ R(\delta_h,\dot\partial_k)\delta_{\bar j}=\widetilde{\Pi}^{\bar i}_{\bar jkh}\delta_{\bar i}\ ,\ R(\delta_h,\dot\partial_{\bar k})\delta_{\bar j}=\widetilde{\Pi}^{\bar i}_{\bar j\bar kh}\delta_{\bar i}\\
&R(\delta_h,\dot\partial_k)\dot\partial_j=\widetilde{P}^{i}_{jkh}\dot\partial_i\ ,\ R(\delta_h,\dot\partial_k)\dot\partial_{\bar j}=\widetilde{P}^{\bar i}_{\bar jkh}\dot\partial_{\bar i}\ ,\ R(\delta_h,\dot\partial_{\bar k})\dot\partial_{\bar j}=\widetilde{P}^{\bar i}_{\bar jkh}\dot\partial_{\bar i}\\
&R(\delta_h,\dot\partial_{\bar k})\delta_j=\widetilde{\Theta}^{i}_{jkh}\delta_i\ ,\ R(\delta_h,\dot\partial_{\bar k})\dot\partial_{\bar j}=\widetilde{Q}^{\bar i}_{\bar jkh}\delta_{\bar i}\\
&R(\dot\partial_h,\dot\partial_k)\delta_j=\widetilde{\Xi}^{i}_{jkh}\delta_i\ ,\ R(\dot\partial_h,\dot\partial_k)\delta_{\bar j}=\widetilde{\Xi}^{\bar i}_{\bar jkh}\delta_{\bar i}\ ,\ R(\dot\partial_h,\dot\partial_{\bar k})\delta_{\bar j}=\widetilde{\Xi}^{\bar i}_{\bar jkh}\delta_{\bar i}\\
&R(\dot\partial_h,\dot\partial_k)\dot\partial_j=\widetilde{S}^{i}_{jkh}\dot\partial_i\ ,\ R(\dot\partial_h,\dot\partial_k)\dot\partial_{\bar j}=\widetilde{S}^{\bar i}_{\bar jkh}\dot\partial_{\bar i}\ ,\ R(\dot\partial_h,\dot\partial_{\bar k})\dot\partial_{\bar j}=\widetilde{S}^{\bar i}_{\bar j\bar kh}\dot\partial_{\bar i}
\end{aligned}
\tag{3.2}
$$

and their conjugates, because $R(X,Y)=-R(Y,X)$.

The local components of curvature tensor R can be obtained easily from (3.1) and (3.2).

Our interest is to compute the curvature tensor K of ∇ by means of R. First, let us decompose the Riemannian curvature K in respect to the adapted base of Chern-Lagrange *c.n.c.*

$$
K(\delta_h,\delta_k)\delta_j=\overset{h}{R}{}^{i}_{jkh}\delta_i+\overset{v}{R}{}^{i}_{jkh}\dot\partial_i+\overset{\bar h}{R}{}^{\bar i}_{jkh}\delta_{\bar i}+\overset{v}{R}{}^{\bar i}_{jkh}\dot\partial_{\bar i}
\tag{3.3}
$$

and similar for $K(\delta_h,\delta_k)\delta_{\bar j}$, $K(\delta_h,\delta_{\bar k})\delta_{\bar j}$, and then for $K(\delta_h,\delta_k)\dot\partial_j$ with coefficients $\overset{h}{\Omega}{}^{i}_{jk}$, etc., at last being $K(\dot\partial_h,\dot\partial_{\bar k})\dot\partial_j$ with coefficients $\overset{h}{S}{}^{i}_{\bar j\bar k}$.., and their conjugates.

Now, we can give directly an expression of Riemannian curvature K as a function of curvature R.

Theorem 3.1. *The local components of curvature of Levi-Civita connection of* $\mathbf{G}$ *metric on* $T_C(T'M)$ *in respect to the adapted base of Chern-Lagrange c.n.c. are:*

$$
\begin{aligned}
&\overset{h}{R}{}^{i}_{jkh}=\widetilde{R}^{i}_{jkh},\ \overset{v}{R}{}^{i}_{jkh}=\overset{\bar h}{R}{}^{\bar i}_{jkh}=\overset{\bar v}{R}{}^{\bar i}_{jkh}=0\\
&\overset{h}{R}{}^{i}_{\bar jkh}=A_{hk}\{\overset{3}{L}{}^{i}_{h\bar j|k}+\overset{2}{D}{}^{l}_{\bar jk}\overset{1}{C}{}^{i}_{hl}+\overset{4}{D}{}^{\bar l}_{\bar jk}\overset{3}{C}{}^{i}_{h\bar l}\},\ \overset{v}{R}{}^{i}_{\bar jkh}=A_{hk}\{\overset{2}{D}{}^{i}_{\bar jh|k}\}\\
&\overset{\bar h}{R}{}^{\bar i}_{\bar jkh}=\widetilde{R}^{\bar i}_{\bar jkh},\ \overset{\bar v}{R}{}^{\bar i}_{\bar jkh}=A_{hk}\{\overset{4}{D}{}^{\bar i}_{\bar jh|k}\},\ \overset{h}{R}{}^{i}_{\bar j\bar kh}=-\{\overset{3}{L}{}^{i}_{h\bar j|\bar k}+\overset{3}{L}{}^{l}_{h\bar k}\overset{3}{L}{}^{i}_{l\bar j}\}\\
&\overset{\bar h}{R}{}^{\bar i}_{\bar j\bar kh}=\widetilde{R}^{\bar i}_{\bar j\bar kh}-\overset{3}{L}{}^{l}_{h\bar j}\overset{3}{L}{}^{\bar i}_{\bar kl}-\overset{3}{C}{}^{\bar i}_{\bar kl}\overset{2}{D}{}^{l}_{\bar jh}-\overset{1}{C}{}^{\bar i}_{\bar k\bar l}\overset{4}{D}{}^{\bar l}_{\bar jh}\\
&\overset{\bar v}{R}{}^{\bar i}_{\bar j\bar kh}=-\overset{4}{D}{}^{\bar i}_{\bar jh|\bar k}-\overset{3}{L}{}^{l}_{h\bar k}\overset{4}{D}{}^{\bar i}_{\bar jl}-\overset{3}{L}{}^{l}_{h\bar j}\overset{2}{D}{}^{\bar i}_{l\bar k}-(\delta_{\bar k}\overset{CL}{N}{}^{l}_{h})\overset{4}{H}{}^{\bar i}_{\bar jl}\\
&\overset{h}{\Omega}{}^{i}_{jkh}=A_{hk}\{\overset{1}{C}{}^{i}_{kj|h}\}\ ,\ \overset{v}{\Omega}{}^{i}_{jkh}=\overset{\bar h}{\Omega}{}^{\bar i}_{jkh}=\overset{\bar v}{\Omega}{}^{\bar i}_{jkh}=0\\
&\overset{h}{\Omega}{}^{i}_{\bar jkh}=A_{hk}\{\overset{3}{C}{}^{i}_{h\bar j|k}+\overset{3}{L}{}^{\bar l}_{\bar jk}\overset{3}{C}{}^{i}_{h\bar l}\}\ ,\ \overset{v}{\Omega}{}^{i}_{\bar jkh}=\overset{\bar h}{\Omega}{}^{\bar i}_{\bar jkh}=\overset{\bar v}{\Omega}{}^{\bar i}_{\bar jkh}=0
\end{aligned}
$$

$$\overset{h}{\Omega}{}^{i}_{\bar{j}kh} = -\overset{3}{C}{}^{i}_{h\bar{j}\overline{|}k} - \overset{3}{L}{}^{l}_{\bar{j}k}\overset{3}{C}{}^{i}_{l\bar{j}} + \overset{1}{C}{}^{\bar{l}}_{k\bar{j}}\overset{3}{L}{}^{i}_{h\bar{l}}\ ,\ \overset{v}{\Omega}{}^{i}_{\bar{j}kh} = \overset{1}{C}{}^{\bar{l}}_{h\bar{j}}\overset{2}{D}{}^{i}_{\bar{l}h} - \overset{3}{C}{}^{l}_{h\bar{j}}\overset{4}{D}{}^{i}_{l\bar{k}}$$

$$\overset{\bar{h}}{\Omega}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \overset{1}{C}{}^{\bar{i}}_{\bar{k}\bar{j}|h} + \overset{3}{L}{}^{\bar{l}}_{\bar{k}h}\overset{1}{C}{}^{\bar{i}}_{\bar{l}\bar{j}} - \overset{3}{C}{}^{l}_{h\bar{j}}\overset{3}{L}{}^{\bar{i}}_{\bar{h}l} + \overline{(\delta_h \overset{CL}{N}{}^{l}_{k})}\overset{1}{G}{}^{\bar{i}}_{\bar{j}\bar{l}}$$

$$\overset{\bar{v}}{\Omega}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \widetilde{\Omega}{}^{\bar{i}}_{\bar{j}\bar{k}h} + \overset{1}{C}{}^{\bar{l}}_{\bar{k}\bar{j}}\overset{4}{D}{}^{\bar{i}}_{\bar{l}h} - \overset{3}{C}{}^{l}_{h\bar{j}}\overset{2}{D}{}^{\bar{i}}_{l\bar{k}}$$

$$\overset{h}{\Pi}{}^{i}_{jkh} = \widetilde{\Pi}{}^{i}_{jkh},\ \overset{v}{\Pi}{}^{i}_{jkh} = \overset{\bar{h}}{\Pi}{}^{\bar{i}}_{jkh} = \overset{\bar{v}}{\Pi}{}^{\bar{i}}_{jkh} = 0\ ,\ \overset{v}{\Pi}{}^{i}_{\bar{j}kh} = -\overset{2}{D}{}^{i}_{\bar{j}h|k} - \overset{1}{C}{}^{l}_{hk}\overset{2}{D}{}^{i}_{\bar{j}l}$$

$$\overset{h}{\Pi}{}^{i}_{\bar{j}kh} = -\overset{3}{L}{}^{i}_{h\bar{j}|k} - \overset{1}{C}{}^{l}_{hk}\overset{3}{L}{}^{i}_{l\bar{j}} + \overset{4}{H}{}^{\bar{l}}_{\bar{j}k}\overset{3}{C}{}^{i}_{h\bar{l}} - \overset{2}{D}{}^{l}_{\bar{j}h}\overset{1}{G}{}^{i}_{lh},$$

$$\overset{\bar{h}}{\Pi}{}^{\bar{i}}{}_{\bar{j}kh} = \widetilde{\Pi}{}^{\bar{i}}_{\bar{j}kh},\ \overset{\bar{v}}{\Pi}{}^{\bar{i}}{}_{\bar{j}kh} = \overset{3}{C}{}^{\bar{l}}_{\bar{j}k}\overset{4}{D}{}^{\bar{i}}_{\bar{l}h} - \overset{3}{L}{}^{\bar{l}}_{\bar{j}h}\overset{4}{H}{}^{\bar{i}}_{\bar{l}k} - (\dot{\partial}_k \overset{KL}{N}{}^{l}_{k})\overset{}{H}{}^{\bar{i}}_{\bar{j}l},$$

$$\overset{h}{\Pi}{}^{i}_{\bar{j}\bar{k}h} = -\overset{3}{L}{}^{i}_{h\bar{j}||k} - \overset{3}{C}{}^{l}_{\bar{j}k}\overset{3}{L}{}^{i}_{l\bar{j}},\ \overset{v}{\Pi}{}^{i}_{\bar{j}\bar{k}h} = -\overset{2}{D}{}^{i}_{\bar{j}h\overline{||}k} - \overset{3}{C}{}^{l}_{h\bar{k}}\overset{2}{D}{}^{i}_{\bar{j}l} - \overset{3}{L}{}^{l}_{h\bar{j}}\overset{4}{\Pi}{}^{i}_{l\bar{k}},$$

$$\overset{\bar{h}}{\Pi}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \widetilde{\Pi}{}^{\bar{i}}_{\bar{j}\bar{k}h} - \overset{4}{D}{}^{\bar{l}}_{\bar{j}h}\overline{\overset{1}{G}{}^{i}_{lkh}},\ \overset{\bar{v}}{\Pi}{}^{\bar{i}}_{\bar{j}\bar{k}h} = -\overset{4}{D}{}^{\bar{i}}_{\bar{j}h\overline{||}k} - \overset{3}{C}{}^{l}_{h\bar{k}}\overset{4}{D}{}^{\bar{i}}_{\bar{j}l} - (\dot{\partial}_{\bar{k}}\overset{CL}{N}{}^{l}_{h})\overset{4}{H}{}^{\bar{i}}_{\bar{j}l},$$

$$\overset{h}{P}{}^{i}_{jkh} = \overset{1}{G}{}^{i}_{jk|h} - \overset{1}{C}{}^{l}_{hj||k} - \overset{1}{C}{}^{l}_{hk}\overset{1}{B}{}^{i}_{jl},\ \overset{v}{P}{}^{i}_{jkh} = \widetilde{P}{}^{i}_{jkh},\ \overset{\bar{h}}{P}{}^{\bar{i}}_{jkh} = \overset{\bar{v}}{P}{}^{\bar{i}}_{jkh} = 0,$$

$$\overset{h}{P}{}^{i}_{\bar{j}kh} = -\overset{3}{C}{}^{i}_{h\bar{j}\overline{||}k} - \overset{1}{C}{}^{l}_{kh}\overset{3}{C}{}^{i}_{l\bar{j}},\ \overset{v}{P}{}^{i}_{\bar{j}kh} = \overset{\bar{h}}{P}{}^{\bar{i}}_{\bar{j}kh} = \overset{\bar{v}}{P}{}^{l}_{\bar{j}kh} = 0,$$

$$\overset{h}{P}{}^{i}_{\bar{j}\bar{k}h} = -\overset{3}{C}{}^{i}_{h\bar{j}\overline{||}k} - \overset{3}{C}{}^{l}_{h\bar{k}}\overset{3}{C}{}^{i}_{l\bar{j}} + \overset{1}{G}{}^{\bar{l}}_{\overline{jk}}\overset{3}{L}{}^{i}_{h\bar{l}},\ \overset{v}{P}{}^{i}_{\bar{j}\bar{k}h} = \overset{1}{G}{}^{l}_{jk}\overset{2}{D}{}^{i}_{lh} - \overset{3}{C}{}^{l}_{h\bar{j}}\overset{4}{H}{}^{i}_{l\bar{k}},$$

$$\overset{\bar{h}}{P}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \overset{1}{G}{}^{i}_{jk|h},\ \overset{\bar{v}}{P}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \widetilde{P}{}^{\bar{i}}_{\bar{j}\bar{k}h} + \overset{1}{G}{}^{l}_{\overline{jk}}\overset{4}{D}{}^{\bar{i}}_{\bar{l}h},\ \overset{h}{\Theta}{}^{i}_{j\bar{k}h} = \widetilde{\Theta}{}^{i}_{j\bar{k}h} + \overset{4}{H}{}^{l}_{j\bar{k}}\overset{1}{C}{}^{i}_{hl},$$

$$\overset{v}{\Theta}{}^{i}_{j\bar{k}h} = \overset{4}{H}{}^{i}_{j\bar{k}|h},\ \overset{\bar{h}}{\Theta}{}^{\bar{i}}_{j\bar{k}h} = \overset{\bar{v}}{\Theta}{}^{\bar{i}}_{j\bar{k}h} = 0,$$

$$\overset{h}{Q}{}^{i}_{j\bar{k}h} = -\overset{1}{C}{}^{i}_{hj\overline{||}k} - \overset{3}{C}{}^{l}_{h\bar{k}}\overset{1}{C}{}^{i}_{lj} - (\dot{\partial}_{\bar{k}}\overset{CL}{N}{}^{l}_{h})\overset{1}{G}{}^{i}_{jl},\ \overset{v}{Q}{}^{i}_{j\bar{k}h} = -\widetilde{Q}{}^{i}_{j\bar{k}h} - \overset{1}{C}{}^{l}_{hj}\overset{4}{H}{}^{i}_{l\bar{k}},$$

$$\overset{\bar{h}}{Q}{}^{\bar{i}}_{j\bar{k}h} = \overset{\bar{v}}{Q}{}^{\bar{i}}_{j\bar{k}h} = 0,\ \overset{h}{\Xi}{}^{i}_{jkh} = \widetilde{\Xi}{}^{i}_{jkh},\ \overset{v}{\Xi}{}^{i}_{jkh} = \overset{\bar{h}}{\Xi}{}^{\bar{i}}_{jkh} = \overset{\bar{v}}{\Xi}{}^{\bar{i}}_{jkh} = 0,$$

$$\overset{h}{\Xi}{}^{i}_{\bar{j}kh} = \overset{v}{\Xi}{}^{i}_{\bar{j}kh} = 0,\ \overset{\bar{v}}{\Xi}{}^{\bar{i}}_{\bar{j}kh} = \widetilde{\Xi}{}^{\bar{i}}_{\bar{j}kh},\ \overset{\bar{v}}{\Xi}{}^{\bar{i}}_{\bar{j}kh} = A_{kh}\{\overset{4}{H}{}^{\bar{i}}_{\bar{j}k||h}\},\ \overset{h}{\Xi}{}^{i}_{\bar{j}\bar{k}h} = \overset{v}{\Xi}{}^{i}_{\bar{j}\bar{k}h} = 0,$$

$$\overset{\bar{h}}{H}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \widetilde{H}{}^{\bar{i}}_{\bar{j}\bar{k}h} - \overset{4}{H}{}^{\bar{l}}_{\bar{j}h}\overset{1}{G}{}^{\bar{i}}_{\overline{lk}},\ \overset{\bar{h}}{H}{}^{\bar{i}}_{\bar{j}\bar{k}h} = -\overset{4}{H}{}^{\bar{i}}_{\bar{j}h\overline{||k}},\ \overset{h}{S}{}^{i}_{jkh} = A_{kh}\{\overset{1}{G}{}^{i}_{jk||h}\},$$

$$\overset{v}{S}{}^{i}_{jkh} = \overset{\bar{h}}{S}{}^{\bar{i}}_{jkh} = \overset{\bar{v}}{S}{}^{\bar{i}}_{jkh} = 0,\ \overset{h}{S}{}^{i}_{\bar{j}kh} = \overset{v}{S}{}^{i}_{\bar{j}kh} = \overset{\bar{h}}{S}{}^{\bar{i}}_{\bar{j}kh} = \overset{\bar{v}}{S}{}^{\bar{i}}_{\bar{j}kh} = 0,\ \overset{h}{S}{}^{i}_{\bar{j}\bar{k}h} = \overset{v}{S}{}^{i}_{\bar{j}\bar{k}h} = 0,$$

$$\overset{\bar{h}}{S}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \overset{1}{G}{}^{\bar{i}}_{\overline{jk}||h},\ \overset{\bar{v}}{S}{}^{\bar{i}}_{\bar{j}\bar{k}h} = \widetilde{S}{}^{\bar{i}}_{\bar{j}\bar{k}h} + \overset{1}{G}{}^{\bar{l}}_{\overline{jk}}\overset{4}{H}{}^{\bar{i}}_{\bar{l}h},$$

and their conjugates, where by $|\,k, ||\,k, \overline{|}k, \overline{||}k$ *we denote* $h-, v-, \bar{h}, \bar{v}-$ *covariant derivatives with respect to* D, *and* $A_{hk}\{T^i_{hk}\} = T^i_{hk} - T^i_{kh}$.

Remark 3.1. If $\delta_k g_{j\bar{l}} = \delta_j g_{k\bar{l}}$, then we shall obtain the particular form of Theorem 3.2.

Although the paper has some technical elements, this calculus is absolutely necessary for a further study of holomorphic sectional curvature.

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