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GENERATION OF NEW EXACT SOLUTIONS FOR A CHARGED RADIATIVE FLUID SPHERE IN GENERAL RELATIVITY

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Abstract. In this paper solutions of the equations of the relativistic charged perfect fluids are analyzed as relativistic models of a radiating balanced sphere. Using a generation technique given in [4], we reduce the field equations for a spherically symmetric distribution of a radiating or charged perfect fluid to a first order nonlinear differential equation of Riccati type.

1 Introduction

The search of new solutions for the field equations is an important aspect of the general relativity theory. The problem of finding exact solutions for Einstein's equations for a static distribution of a fluid sphere in the presence of the radiation or electromagnetic field is one of distinct interest. It is suggested that the collapse of a spherically symmetric distribution of matter to a point singularity can be avoided if the matter is accompanied by such fields. Solutions of the Einstein field equations for a perfect fluid with or without a radiation or electromagnetic field have been studied by several authors [1]–[9].

In this paper, the difficulty owing to the nonlinearity of the field equations has been overcome by using a generation technique given in [4]. This method reduces the field equations in the case of spherically symmetric distributions of a radiating or charged perfect fluid to a first order nonlinear differential equation of Riccati type. If a particular solution is known, this equation can be reduced to a linear first order differential equation.

2 Field equations for a charged radiative fluid

In comoving coordinates in which we choose the units so that $c = 1$, the line element of the space-time will be taken in the form

$$ds^2 = e^{2\varphi}dt^2 - e^{2\psi}dr^2 - r^2 d\Omega^2, \tag{2.1}$$

where $\varphi = \varphi(r)$, $\psi = \psi(r)$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\chi^2$, θ and χ labelling points on the unit sphere. The functions φ and ψ satisfy the Einstein equations

$$G^\alpha_\beta = \varkappa T^\alpha_\beta, \tag{2.2}$$

and the conservation identities

$$T^{\alpha\beta}_{;\beta} = 0, \tag{2.3}$$

where the energy-momentum tensor T^{α}_{β} of the relativistic thermodynamical perfect fluid [10]–[11] is the sum of the fluid energy-momentum tensor, the energy-momentum tensor of the radiation field and the energy-momentum tensor of the electromagnetic field

$$T^{\alpha}_{\beta} = wu^{\alpha}u_{\beta} - p \overset{\perp}{g}{}^{\alpha}_{\beta} + Q^{\alpha}_{\beta} + E^{\alpha}_{\beta}, \quad \alpha, \beta = 0, 1, 2, 3, \tag{2.4}$$

where $\overset{\perp}{g}{}^{\alpha}_{\beta} = g^{\alpha}_{\beta} - u^{\alpha}u_{\beta}$ is the spatial projector, $u_{\alpha} = (e^{\varphi}, 0, 0, 0)$ is the fluid velocity, with

$$Q^{\alpha}_{\beta} = Qu^{\alpha}u_{\beta} + u^{\alpha}q_{\beta} + q^{\alpha}u_{\beta} - \frac{1}{3}Q \overset{\perp}{g}{}^{\alpha}_{\beta}, \tag{2.5}$$

where Q is the density of radiation energy and $q_{\alpha} = \left(0, -qe^{\psi}, 0, 0\right)$ is the radiative flux and verifies the equations of radiation field

$$Q^{\alpha\beta}_{;\beta} + F^{\alpha} = 0,$$

with $F^{\alpha} = \sigma\left(q^{\alpha} + \left(Q - aT^4\right)u^{\alpha}\right)$, and

$$E^{\alpha}_{\beta} = F_{\alpha\gamma}F^{\beta\gamma} - \frac{1}{4}g^{\beta}_{\alpha}F_{\gamma\delta}F^{\gamma\delta} \tag{2.6}$$

where, the tensor $F_{\alpha\beta}$ satisfies the Maxwell equations

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \quad F^{\alpha\beta}_{;\beta} = J^{\alpha}.$$

The fluid will be assumed to have null conductivity, so that

$$J^{\alpha} = \sigma u^{\alpha}.$$

3 Generation of new solutions for spherically distribution

3.1 The case of a radiative fluid sphere

In this case, the Einstein equations and the conservation identities with $E^{\alpha}_{\beta} = 0$, become:

$$1 - (1 - 2r\psi')\,e^{-2\psi} = \varkappa r^2 w^*, \tag{3.1}$$

$$1 - (1 + 2r\varphi')\,e^{-2\psi} = -\varkappa r^2 p^*, \tag{3.2}$$

$$\left(\varphi'' + (\varphi')^2 + \frac{1}{r}(\varphi' - \psi') - \varphi'\psi'\right)e^{-2\psi} = \varkappa p^*, \tag{3.3}$$

$$Q = aT^4, \quad q = 0, \tag{3.4}$$

$$(p^*)' + (p^* + w^*)\,\varphi' = 0, \tag{3.5}$$

where the prime denotes differentiation with respect to r. The other equations being identically satisfied. Here, $w^* = w + Q$, $p^* = p + \frac{1}{3}Q$. From (3.4) it results that the static solutions can be found only in the case of a fluid in local thermodynamic equilibrium.

A first integral of the equation (3.1) is [1]

$$e^{-2\psi(r)} = 1 - \frac{2m(r)}{r}, \quad 2m(r) = \varkappa \int r^2 w^*(r)\, dr. \tag{3.6}$$

Inserting this into (3.2), one obtains

$$2r(r-2m)\varphi' = \varkappa r^3 p^* + 2m.$$

Eliminating φ_r by means of (3.5), one gets

$$2r(2m-r)(p^*)' = \varkappa r^3 (p^*)^2 + \left(2m + \varkappa r^3 w^*\right) p^* + 2mw^*, \tag{3.7}$$

which is a Riccati equation in p^*, whose coefficients depend on w^*.

If a solution (w^*, p_0^*) is known, a new solution (w^*, p^*) can be generated by

$$p^* = p_0^* + \frac{1}{y},$$

where y is the solution of linear equation

$$2r(2m-r)y' + \left[2m + \varkappa r^3 (2p_0^* + w^*)\right] y + \varkappa r^3 = 0.$$

Once a solution (w^*, p^*) is known, φ and ψ can be computed from (3.5) and (3.6).

A different form of the field equations can be obtained eliminating p^* from (3.2) and (3.3). One gets

$$\varphi'' + (\varphi')^2 - \left(\psi' + \frac{1}{r}\right)\varphi' - \frac{1}{r}\left(\psi' + \frac{1}{r}\right) + \frac{1}{r^2}e^{2\psi} = 0, \tag{3.8}$$

The equation (3.8) is a Riccati equation in $\Phi = \varphi'$, whose coefficients depend on ψ,

$$\Phi' + \Phi^2 - \left(\psi' + \frac{1}{r}\right)\Phi - \frac{1}{r}\left(\psi' + \frac{1}{r}\right) + \frac{1}{r^2}e^{2\psi} = 0. \tag{3.9}$$

If (Φ_0, ψ) is a solution of the equation (3.9), then a new solution (Φ, ψ) can be generated by

$$\Phi = \Phi_0 + \frac{1}{x}, \tag{3.10}$$

where x satisfies the linear equation

$$x' + \left(\psi' + \frac{1}{r} - 2\Phi_0\right)x - 1 = 0. \tag{3.11}$$

Because $\Phi_0 = \varphi_0'$, the solution of the equation (3.11) is

$$x(r) = \frac{1}{r}e^{-\psi + 2\varphi_0}\left(C + \int r e^{\psi - 2\varphi_0} dr\right).$$

Putting

$$F(r,C) = C + \int re^{\psi - 2\varphi_0} dr,$$

from (3.10) one obtains

$$(\varphi - \varphi_0)' = \frac{F'}{F},$$

such that

$$e^{\varphi(r)} = C_0 e^{\varphi_0(r)} \cdot F(r,C), \tag{3.12}$$

where C and C_0 are arbitrary constants.

If a solution (φ_0, ψ) is known, using (3.12) a new solution (φ, ψ) can be generated.

Once the metric functions φ, ψ are known, w^* and p^* can be computed from (3.1) and (3.2).

3.2 The case of a charged radiative fluid sphere

In this case, the Einstein equations and the conservation identities become:

$$1 - (1 - 2r\psi') e^{-2\psi} = \varkappa r^2 (w^* + k), \tag{3.13}$$

$$1 - (1 + 2r\varphi') e^{-2\psi} = -\varkappa r^2 (p^* - k), \tag{3.14}$$

$$\left(\varphi'' + (\varphi')^2 + \frac{1}{r}(\varphi' - \psi') - \varphi'\psi'\right) e^{-2\psi} = \varkappa (p^* + k), \tag{3.15}$$

where

$$k = \frac{1}{2} F_{01} F^{01}.$$

We assume an equation of state of the form

$$p^* = (\gamma - 1) w^*. \tag{3.16}$$

From the equations (3.14) and (3.15) we find:

$$\left(\varphi'' + \frac{2}{r}\varphi' + (\varphi')^2 + \frac{\varphi' - \psi'}{r} - \varphi'\psi' + \frac{1}{r^2}\right) e^{-2\psi} - \frac{1}{r^2} = 2\varkappa p^*, \tag{3.17}$$

$$\left(\varphi'' + (\varphi')^2 - \frac{\varphi' + \psi'}{r} - \varphi'\psi' - \frac{1}{r^2}\right) e^{-2\psi} + \frac{1}{r^2} = 2\varkappa k. \tag{3.18}$$

Equations (3.13) and (3.14) lead to

$$2\frac{\varphi' + \psi'}{r} e^{-2\psi} = \varkappa (p^* + w^*),$$

or, with (3.16)

$$2\frac{\varphi' + \psi'}{r} e^{-2\psi} = \varkappa \gamma w^*. \tag{3.19}$$

Using (3.16) and (3.19), equation (3.17) can be written

$$\varphi'' + (\varphi')^2 - \left(\frac{\gamma - 4}{\gamma r} + \psi'\right)\varphi' - \frac{3\gamma - 4}{\gamma r}\psi' + \frac{1}{r^2}\left(1 - e^{2\psi}\right) = 0,$$

which is a Riccati equation in $\Phi = \varphi'$, whose coefficients depend on ψ,

$$\Phi' + \Phi^2 - \left(\frac{\gamma - 4}{\gamma r} + \psi'\right)\Phi - \frac{3\gamma - 4}{\gamma r}\psi' + \frac{1}{r^2}\left(1 - e^{2\psi}\right) = 0. \tag{3.20}$$

If (Φ_0, ψ) is a solution of the equation (3.20), then a new solution (Φ, ψ) can be generated by

$$\Phi = \Phi_0 + \frac{1}{z}, \tag{3.21}$$

where z satisfies the linear equation

$$z' + \left[\psi' + \left(1 - \frac{4}{\gamma}\right)\frac{1}{r} - 2\Phi_0\right] z - 1 = 0. \tag{3.22}$$

whose general solution is

$$z(r) = r^{1-\frac{4}{\gamma}} e^{-\psi + 2\varphi_0}\left(C + \int r^{\frac{4}{\gamma}-1} e^{\psi - 2\varphi_0} dr\right).$$

If we now set

$$G(r, C) = C + \int r^{\frac{4}{\gamma}-1} e^{\psi - 2\varphi_0} dr,$$

one obtains

$$e^{\varphi(r)} = C_0 e^{\varphi_0(r)} \cdot G(r, C), \tag{3.23}$$

where C and C_0 are arbitrary constants.

If a solution (φ_0, ψ) is known, using (3.23), a new solution (φ, ψ) can be generated.

Once the metric functions φ, ψ are known, w^*, p^* and k can be computed from (3.16), (3.17) and (3.18).

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