LIBERTAS MATHEMATICA, VOL XXII (2002)

NON-STEADY SHEARING FLOWS OF A MAXWELL FLUID

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Abstract. The velocity fields and the associated tangential tensions corresponding to a Maxwell fluid subject to a shearing flow between two infinite parallel plates are determined in two different cases. The similar solutions corresponding to a Navier-Stokes fluid as well as those of the steady state appear as limiting cases of our solutions.

1 Introduction

Maxwell models are quite useful in the study of dilute polymeric fluids. They are especially computable for small dimensionless relaxation times [1]. However, there are situations when such a model is applied to viscoelastic problems where the dimensionless relaxation time is large, as would be the case of more concentrated polymeric fluids [2].

The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is related to the fluid motion by [1] (see also (4.3) of [3] with $\lambda_2 = 0$)

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\top}) = 2\mu\mathbf{D}; \quad \operatorname{tr}\mathbf{D} = 0, \tag{1.1}$$

where $\mathbf{S}$ is the extra-stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{L}$ is the velocity gradient, $\mathbf{D}$ is the rate of the strain tensor, λ is the relaxation time, μ the dynamic viscosity and the superposed dot denotes the material time derivative.

The aim of this note is to present the velocity fields and the associated tangential tensions corresponding to two shearing flows of an incompressible Maxell fluid. These solutions satisfy all imposed initial and boundary conditions and for $\lambda \to 0$ they are going to those corresponding to a Navier-Stokes fluid. The steady state solutions appear also as limiting the cases for $t \to \infty$.

2 Statement of the problem

Consider an incompressible Maxwell fluid, at rest, lying between two infinite parallel plates. At time $t = 0^+$ the lower plate begins to move with the constant velocity V in a direction parallel to the upper one which is stationary. In a suitable Cartesian co-ordinate system the velocity field has the form

$$\mathbf{v} = (v(y, t), 0, 0). \tag{2.1}$$

For such a flow, which is named the rectilinear shearing flow, the constraint of incompressibility is automatically satisfied and the stress field is independent of x and z. Equation $(1.1)_2$ together with the natural initial condition $\mathbf{S}(y,0)=\mathbf{0}$ imply $S_{yy}=S_{yz}=S_{zz}=S_{zx}=0$ and

$$(1+\lambda\partial_t)\tau(y,t)=\mu\partial_y V(y,t), \qquad (1+\lambda\partial_t)\sigma(y,t)=2\tau(y,t)\partial_y v(y,t), \tag{2.2}$$

where $\tau=S_{xy}$ is the tangential tension and $\sigma=S_{xx}$.

The equations of motion, in the absence of the pressure gradient in the x-direction, reduce to

$$\partial_y\tau(x,t)=\rho\partial_t v(y,t), \tag{2.3}$$

where ρ is the constant density of the fluid.

Eliminating τ between $(2.2)_1$ and (2.3) we attain to the linear partial differential equation

$$\lambda\partial_t^2 v(y,t)+\partial_t v(y,t)=\nu\partial_y^2 v(y,t); \qquad 0<y<h,\ t>0, \tag{2.4}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid and h the distance between the two plates. Assuming that the fluid adheres to the walls, we have the boundary conditions

$$v(0,t)=V, \quad v(h,t)=0; \qquad t>0, \tag{2.5}$$

as well as, the initial conditions (cf. [4])

$$v(y,0)=\partial_t v(y,0)=0; \qquad -\leq y\leq h. \tag{2.6}$$

3 Solution of the problem

3.1. The velocity field. Multiplying both sides of equation (2.4) by $\sin(\lambda_n y)$, integrating between the limits $y=0$ and $y=h$ and taking into account (2.5) and (2.6), we find that

$$\lambda\ddot{v}_n(t)+\dot{v}_n(t)+\nu\lambda_n^2 v_n(t)=a_n; \qquad v_n(0)=\dot{v}_n(0)=0, \tag{3.1}$$

where $a_n=\nu V\lambda_n$, $\lambda_n=n\pi/h$ and $v_n(\cdot)$ is the finite Fourier sine transform of the function $v(y,\cdot)$.

Equation $(3.1)_1$, solved under the initial conditions $(3.1)_{2,3}$ gives

$$v_n(t)=\begin{cases}\dfrac{V}{\lambda_n}\left[1-\dfrac{r_{1n}\exp(r_{2n}t)-r_{2n}\exp(r_{1n}t)}{r_{1n}-r_{2n}}\right] & \text{for } n\leq m,\\[2ex] \dfrac{V}{\lambda_n}\left\{1-\exp\left(-\dfrac{t}{2\lambda}\right)\left[\cos\left(\dfrac{\beta_n t}{2\lambda}\right)+\dfrac{1}{\beta_n}\sin\left(\dfrac{\beta_n t}{2\lambda}\right)\right]\right\} & \text{for } n>m,\end{cases} \tag{3.2}$$

where $r_{1n}, r_{2n}=\dfrac{-1\pm\sqrt{1-4\nu\lambda\lambda_n^2}}{2\lambda}$, $\beta_n=\sqrt{4\nu\lambda\lambda_n^2-1}$ and $m=\left[\dfrac{h}{2\pi\sqrt{\nu\lambda}}\right]$ is the integer part of $h/(2\pi\sqrt{\nu\lambda})$.

Inverting this result by means of Fourier's sine formula [5] we have

$$v(y,t) = V\left(1-\frac{y}{h}\right) - \frac{2V}{h}\sum_{n=1}^{m}\frac{r_{1n}\exp(r_{2n}t) - r_{2n}\exp(r_{1n}t)}{r_{1n}-r_{2n}}\frac{\sin(\lambda_n t)}{\lambda_n} -$$

$$-\frac{2V}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=m+1}^{\infty}\left[\cos\left(\frac{\beta_n t}{2\lambda}\right) + \frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]. \tag{3.3}$$

3.2. The tangential tension. The solution of the ordinary differential equation $(2.2)_1$ with the initial condition $\tau(y, 0 = 0$ is of the form

$$\tau(y,t) = \frac{\mu}{\lambda}\int_0^t \exp\left(\frac{\tau - t}{\lambda}\right)\partial_y v(y,t)d\tau. \tag{3.4}$$

Introducing (3.3) in (3.4) we attain to

$$\tau(y,t) = \frac{\mu V}{h}\left[\exp\left(-\frac{t}{\lambda}\right) - 1\right] + \frac{2\mu V}{h}\sum_{n=1}^{m}\frac{\exp(r_{2n}t) - \exp(r_{1n}t)}{\sqrt{1-4\nu\lambda\lambda_n^2}}\cos(\lambda_n y) -$$

$$-\frac{4\mu V}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=m+1}^{\infty}\frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\cos(\lambda_n y). \tag{3.5}$$

Finally, the normal stress $\sigma(y,t)$ can be also determined from $(2.2)_2$, (3.3), (3.5) and the initial condition $\sigma(y, 0) = 0$.

4 Flow due to a constant pressure gradient

Let us now consider a constant pressure gradient applied at time $t = 0^+$ to a Maxwell fluid contained between two infinite parallel plates at rest. In this case the linear partial differential equation (2.4) and the boundary conditions (2.5) become (see also [4] where $\partial p/\partial x = -A\rho$)

$$\lambda\partial_t^2 v(y,t) + \partial_t(y,t) = A + \nu\partial_y^2 v(y,t); \qquad 0 < y < h,\ t > 0, \tag{4.1}$$

respectively

$$v(0,t) = v(h,t) = 0; \qquad t > 0. \tag{4.2}$$

Following the same way as before we get (see [5], the entries 1 and 8 of Table IX) for $v(y,t)$ and $\tau(y,t)$ the expressions:

$$v(y,t) = \frac{Ay(h-y)}{2\nu} - \frac{4A}{\nu h}\sum_{n=1}^{m}\frac{r_{1n}\exp(r_{2n}t) - r_{2n}(r_{1n}t)}{r_{1n}-r_{2n}}\frac{\sin(\lambda_n y)}{\lambda_n^3} -$$

$$-\frac{4A}{\nu h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=m+1}^{\infty}\left[\cos\left(\frac{\beta_n t}{2\lambda}\right) + \frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\right]\frac{\sin(\lambda_n y)}{\lambda_n^3}, \tag{4.3}$$

respectively,

$$\tau(y,t) = \frac{\rho A}{2}(h-2y)\left[1-\exp\left(-\frac{t}{\lambda}\right)\right] + \frac{4\rho A}{h}\sum_{n=1}^{m}\frac{\exp(r_{2n}t)-\exp(r_{1n}t)}{\sqrt{1-4\nu\lambda\lambda_n^2}}\frac{\cos(\lambda_n y)}{\lambda_n^2} -$$
$$-\frac{4\rho A}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=m+1}^{\infty}\frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right)\frac{\cos(\lambda_n y)}{\lambda_n^2}. \tag{4.4}$$

where the sums are taken only for odd values of n.

Direct computations show that $v(y,t)$ and $\tau(y,t)$ satisfy, in each case, both the associate partial differential equations and all imposed initial and boundary conditions, the differentiation term by term in y and t being clearly permissible.

5 Limiting cases

Taking the limits of Eqs. (3.3), (3.5), (4.3) and (4.4) as $\lambda \to 0$, we obtain the similar solutions corresponding to a Navier-Stokes fluid (see [6], eq. (9.5) where the method of separation of variables was used)

$$v(y,t) = V\left(1-\frac{y}{h}\right) - \frac{2V}{h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n}\exp(-\nu\lambda_n^2 t), \tag{5.1}$$

$$\tau(y,t) = -\frac{\mu V}{h} - \frac{2\mu V}{h}\sum_{n=1}^{\infty}\cos(\lambda_n t)\exp(-\nu\lambda_n^2 t), \tag{5.2}$$

$$v(y,t) = \frac{Ay(h-y)}{2\nu} - \frac{4A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_{2n-1}y)}{\lambda_{2n-1}^3}\exp(-\nu\lambda_{2n-1}^2 t) \tag{5.3}$$

and (eq. (5.3) is in accordance with the last relation of [7], §4)

$$tau(y,t) = \frac{\rho A}{2}(h-2y) - \frac{4\rho A}{h}\sum_{n=1}^{\infty}\frac{\cos(\lambda_{2n-1}y)}{\lambda_{2n-1}^2}\exp(-\nu\lambda_{2n-1}^2 t), \tag{5.4}$$

Remark. The expressions of the velocity and tension fields corresponding to a Maxwell fluid, contains sine and cosine terms in t, while such terms does not appear in the case of Navier-Stokes fluid. This indicates that in non-steady shearing flows of such liquids oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$ or $\exp(-t/\lambda)$.

By letting now $t \to \infty$ in anyone of the above expressions, we get the corresponding solutions for the steady state flow

$$v(y) = V(1-y/h), \qquad \tau = -\mu V/h \tag{5.5}$$

and

$$v(y) = Ay(h-y)/(2\nu), \qquad \tau = \rho A(h-2y)/2. \tag{5.6}$$

Consequently, the velocity fields and the associated tangential tensions corresponding to the steady state are the same for both types of fluid, Newtonian or not.

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