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NONPERIODIC WAVES IN PIEZOELECTRICITY

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Abstract. In this paper we consider the dynamic ally coupled electromechanical responses of a linear, homogeneous and isotropic piezoelectric material which occupied a half-space. Here we consider a particular problem in which the boundary and initial conditions are such that the mechanical problem is one dimensional. The studied problem is based on Mindlin's model. The obtained results cleary emphasize the interaction between the electrical and mechanical fields, and the "softening" effect for these materials.

1 Introduction

Piezoelectricity describes the phenomen of the generation of an electric charge in a material when subjected to a mechanical stress, the so-called direct effect, and conversely, a mechanical strain with response to an applied electric field, the converse effect.

We apply in this contribution, Mindlin's linear model [4], in the quasielectrostatic approximation for the electromagifield. This model, by incorporating — besides strain and electric polarization — the gradient of the electric polarization into the internal energy density, takes into account more precisely the microscopic aspects of structure and interatomic interactions.

This article obtains solutions for displacement, polarization and electric field generated by a mechanical stress shock on the boundary of the centrosymmetric, isotropic, elastic dielectric half-space. In this paper the Laplace transform on time of the governing equations of one–dimensional piezoelectricity are considered together with the zero initial conditions.

It is of interest to note that, depending on the sequence in which the solutions to the final equations governing either the transformed displacement or the polarization are substituted in the governing field equations, two different forms of the solution are obtained. It is shown here that these two forms are equivalent.

We obtain the inverse Laplace transforms for the transformed displacement, the polarization and the electric potential. The obtained ordinary differential equation in the transformed displacement or polarization have the differential operator identical with the differential obtained by using the Helmholtz's decomposition [1].

It is shown that the displacement, polarization and electric potential are discontinuous functions.

2 Governing Equations

We refer the motion of continuum to a fixed system of rectangular Cartesian axes. For a homogeneous, isotropic and centrosymmetric body occupying the region D, with the boundary ∂D, the basic equations in the linear theory of quasielectrostatic piezoelectricity, in the absence of an external body force and an external electric field, are in [1], [4].

The equations of motion and electric field are

$$\begin{aligned} & c_{44}\nabla^2\mathbf{u} + (c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u} + d_{44}\nabla^2\mathbf{P} + (d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P} = \rho\ddot{\mathbf{u}}, \\ & d_{44}\nabla^2\mathbf{u} + (d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u} + (b_{44}+b_{77})\nabla\nabla\mathbf{u} + (b_{44}+b_{77})\nabla^2\cdot\mathbf{P} + \\ & \qquad + (b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P} - a\mathbf{P} - \nabla\varphi = 0, \\ & -\varepsilon_0\nabla^2\varphi + \nabla\cdot\mathbf{P} = 0, \quad \text{in } D. \end{aligned} \tag{1}$$

$$\nabla^2\varphi = 0, \quad \text{in vacuum.} \tag{1'}$$

The constitutive equations are given by

$$\begin{aligned} & t_{ij} = c_{12}u_{k,k}\delta_{ij} + c_{44}(u_{i,j}+u_{j,i}) + d_{12}P_{k,k}\delta_{ij} + d_{44}(P_{j,i}P_{i,j}), \\ & \pi_{ij} = d_{12}u_{k,k}\delta_{ij} + d_{44}(u_{i,j}+u_{j,i}) + b_{12}P_{k,k}\delta_{ij} + b_{44}(P_{j,i}+P_{i,j}) + \\ & \qquad + b_{77}(P_{j,i}-P_{i,j}) + b^0\delta_{ij}, \\ & \pi_i = -aP_i. \end{aligned} \tag{2}$$

The geometrical equations are

$$e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \tag{3}$$

In the preceding formulas, $\mathbf{u}$ denotes the displacement vector field, $\mathbf{P}$ is the polarization vector field, φ is the electric potential, t_{ij} – the components of the stress tensor, π_i – the components of the effective local electric force, π_{ij} – the components of electric tensor, ρ, c_{12}, c_{44}, d_{12}, d_{44}, b_{12}, b_{44}, b_{77}, a, b^0 are the material constants and ε_0 is the permitivity of the vacuum.

Let the piezoelectric half-space be defined by $0 < x_1 < \infty$, $-\infty < x_2, x_3 < \infty$ with the boundary $x_1 = 0$.

The boundary and the initial conditions for the considered problem are

$$\begin{aligned} & t_{11}(0,t) = f_0 H(t), \\ & t_{1j}^{(0,t)} = t_{j1}^{(0,t)} = 0, \quad t_{j2}^{(0,t)} = t_{j3}^{(0,t)} = 0, \qquad j = 2,3 \end{aligned} \tag{4}$$

$$\pi_{ij}(0,t) = 0, \quad i,j = 1,2,3, \tag{5}$$

$$u_i(\mathrm{x},0) = 0, \quad \dot{u}_i(\mathrm{x},0) = 0, \tag{6}$$

where $H(t)$ is the Heaviside's function.

Since the problem is essentially one-dimensional, only $u_1(x_1,t)$, $P_1(x_1,t)$ and $\varphi(x_1,t)$ are nonvanishing.

The basic equations (1) and (2) are reduced to

$$\begin{gathered}(2c_{44}+c_{12})u_{1,11}+(2d_{44}+d_{12})P_{1,11}=\rho\ddot{u}_1,\\ (2d_{44}+d_{12})u_{1,11}+(2b_{44}+b_{12})P_{1,11}-aP_1-\varphi_{,1}=0,\\ -\varepsilon_0\varphi_{,11}+P_{1,1}=0,\end{gathered}\tag{7}$$

$$\begin{gathered}t_{11}=a_1u_{1,1}+a_2P_{1,1},\quad t_{1j}=t_{j1}=0,\quad j=2,3\\ t_{22}=t_{33}=c_{12}u_{1,1}+d_{12}P_{1,1},\quad t_{23}=t_{32}=0\end{gathered}\tag{8}$$

$$\begin{gathered}\pi_{11}=a_2u_{1,1}+a_3P_{1,1}+b^0,\quad \pi_{1j}=\pi_{j1}=0,\quad j=2,3\\ \pi_{22}=\pi_{33}=d_{12}u_{1,1}+b_{12}P_{1,1}+b^0,\quad \pi_{23}=\pi_{32}=0.\end{gathered}$$

where

$$a_1=2c_{44}+c_{12},\quad a_2=2d_{44}+d_{12},\quad a_3=2b_{44}+b_{12}.\tag{9}$$

From $(7)_3$ we get

$$\varphi_{,11}=\frac{1}{\varepsilon_0}P_{1,1}\tag{10}$$

and the system (7) can be rewritten as

$$\begin{cases}a_1u_{1,11}+a_2P_{1,11}=\rho\ddot{u}_1\\ a_2u_{1,111}+a_3P_{1,111}-a_4P_{1,1}=0\end{cases}\tag{11}$$

where $a_4=a+\varepsilon_0^{-1}$.

3 The Development of the Method

A convenient method for solving the problem is to take the Laplace transform on time of the reduced equations and boundary conditions, where the Laplace transform is defined as

$$\overline{F}(x_1,s)=\int_0^\infty F(x_1,t)e^{-st}\,\mathrm{d}\,t.$$

Performing the Laplace transform in (11), we obtain

$$\begin{gathered}a_1\overline{u}_{1,11}+a_2\overline{P}_{1,11}=\rho s^2\overline{u}_1,\\ a_2\overline{u}_{1,111}+a_3\overline{P}_{1,111}-a_4\overline{P}_{1,1}=0.\end{gathered}\tag{12}$$

From (12) we get, that in the domain of the transforms, the fields $\overline{u}_1(x_1, s)$ and $\overline{P}_1(x_1, s)$ are the solutions of the following equation

$$\left[(a_2^2 - a_1 a_3)\frac{\partial^4}{\partial x_1^4} + (\rho a_3 s^2 + a_1 a_4)\frac{\partial^2}{\partial x_1^2} - \rho a_4 s^2\right](\overline{u}_1 \overline{P}_1) = 0. \tag{13}$$

The characteristic equation corresponding to Eq. (13) is given by

$$(a_2^2 - a_1 a_3)k^4 + (\rho a_3 s^2 + a_1 a_4)k^2 - \rho a_4 s^2 = 0, \tag{14}$$

with the roots

$$k_{1,2}^2 = \frac{1}{2(a_2^2 - a_1 a_3)}\left[-(\rho a_3 s^2 + a_1 a_4) \pm \sqrt{\rho^2 a_3^2 s^4 + 2\rho a_4(2a_2^2 - a_1 a_3)s^2 + a_1^2 a_4^2}\right]. \tag{15}$$

The solutions of Eq. (13), bounded at infinity, are

$$\overline{u}_1(x_1, s) = Ae^{-k_1 x_1} + Be^{-k_2 x_1}, \tag{16}$$

$$\overline{P}_1(x_1, s) = Ce^{-k_1 x_1} + De^{-k_2 x_1}, \tag{17}$$

where only two of the constants of integration A, B, C, D that depend on the transform parameter s are independent.

Two distinct but equivalent forms of the solution are derived alternatively by substituting Eq. (16) and (17) for either Eq. $(12)_1$ or Eq. $(12)_2$.

4 The First Form of the Solution in the Domain of the Transforms

Substituting Eqs. (16) and (17) into Eq. $(12)_1$, we have

$$A = \frac{a_4 - a_3 k_1^2}{a_2 k_1^2}C, \qquad B = \frac{a_4 - a_3 k_2^2}{a_2 k_2^2}D. \tag{18}$$

The values given by Eqs.(18) are now inserted into Eq.(16) to obtain

$$\overline{u}_1(x_1, s) = \frac{a_4 - a_3 k_1^2}{a_2 k_1^2}Ce^{-k_1 x_1} + \frac{a_4 - a_3 k_2^2}{a_2 k_2^2}De^{-k_1 x_1}. \tag{19}$$

The two constants of integration in Eqs.(17) and (19) can now determined from the Laplace transform of the boundary conditions. Firstly we consider the following general conditions

$$t_{11}(0, t) = f(t), \qquad \pi_{11}(0, t) = 0. \tag{20}$$

From (8), (17), (19) and (20) we get the first form of the solution as

$$\begin{aligned}\overline{u}_1(x_1, s) = {} & \frac{a_4 - a_3 k_1^2}{(a_1 a_3 - a_2^2)(k_1^2 - k_2^2)}\left\{\frac{1}{k_1}\overline{f}(s) - \frac{b^0\left[(a_1 a_3 - a_2^2)k_2^2 - a_1 a_4\right]}{s k_1 a_2 a_4}\right\}e^{-k_1 x_1} + \\ & + \frac{a_4 - a_3 k_2^2}{(a_1 a_3 - a_2^2)(k_1^2 - k_2^2)}\left\{-\frac{1}{k_2}\overline{f}(s) + \frac{b^0\left[(a_1 a_3 - a_2^2)k_1^2 - a_1 a_4\right]}{s k_2 a_2 a_4}\right\}e^{-k_2 x_1},\end{aligned} \tag{21}$$

$$\begin{aligned}\overline{P}_1(x_1,s) &= \frac{k_1}{(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left\{a_2\overline{f}(s) - \frac{b^0\left[(a_1a_3-a_2^2)k_2^2-a_1a_4\right]}{sa_4}\right\}e^{-k_1x_1}+\\ &+\frac{k_2}{(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left\{-a_2\overline{f}(s) + \frac{b^0\left[(a_1a_3-a_2^2)k_1^2-a_1a_4\right]}{sa_4}\right\}e^{-k_2x_1}.\end{aligned} \tag{22}$$

5 The Second Form of the Solution in the Domain of the Transforms

Substituting Eqs. (16) and (17) into Eq. $(12)_2$ we have

$$C = -\frac{a_1k_1^2-\rho s^2}{a_2k_1^2}A, \qquad D = -\frac{a_1k_2^2-\rho s^2}{a_2k_2^2}B. \tag{23}$$

From (17) and (23) it follows

$$\overline{P}_1(x_1,s) = -\frac{a_1k_1^2-\rho s^2}{a_2k_1^2}Ae^{-k_1x_1} - \frac{a_1k_2^2-\rho s^2}{a_2k_2^2}Be^{-k_2x_1}. \tag{24}$$

By using Eqs. (8), (16), (20) and (24), we get the second form of the solution as

$$\begin{aligned}\overline{u}_1(x_1,s) &= \frac{k_1}{(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left[-\frac{a_2b^0}{s} + \frac{(a_1a_3-a_2^2)k_2^2-a_3\rho s^2}{\rho s^2}\overline{f}(s)\right]e^{-k_1x_1}+\\ &+\frac{k_2}{(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left[\frac{a_2b^0}{s} - \frac{(a_1a_3-a_2^2)k_1^2-a_3\rho s^2}{\rho s^2}\overline{f}(s)\right]e^{-k_2x_1},\end{aligned} \tag{25}$$

$$\begin{aligned}\overline{P}_1(x_1,s) &= \frac{a_1k_1^2-\rho s^2}{s(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left[\frac{b^0}{k_1} - \frac{(a_1a_3-a_2^2)k_2^2-a_3\rho s^2}{a_2\rho k_1 s}\overline{f}(s)\right]e^{-k_1x_1}+\\ &+\frac{a_1k_2^2-\rho s^2}{s(a_1a_3-a_2^2)(k_1^2-k_2^2)}\left[-\frac{b^0}{k_2} + \frac{(a_1a_3-a_2^2)k_1^2-a_3\rho s^2}{a_2\rho k_2 s}\overline{f}(s)\right]e^{-k_2x_1}.\end{aligned} \tag{26}$$

6 The Equivalence of the Two Forms of the Solution

The characteristic equation given by expression in Eq. (14) shows that the sum of the squares of the roots is

$$k_1^2+k_2^2 = -\frac{\rho a_3s^2+a_1a_4}{a_2^2-a_1a_3}, \tag{27}$$

and that their product is

$$k_1^2k_2^2 = -\frac{\rho a_4s^2}{a_2^2-a_1a_3}. \tag{28}$$

The equivalence of these two different forms of the solutions is established by using the relations (23)-(28) and a straightforward calculation.

7 The Solution of the Problem

The roots of the characteristic equation (14) can be written in the following form:

$$\begin{aligned} k_1', k_1'' &= \pm m_0 \left[\sqrt{p^2+m_2^2} - \sqrt{(p-2m_1)^2+m_2^2}\right], \\ k_2', k_2'' &= \pm m_0 \left[\sqrt{p^2+m_2^2} + \sqrt{(p-2m_1)^2+m_2^2}\right], \end{aligned} \tag{29}$$

where

$$m_0 = \sqrt{\frac{\rho a_3}{4(a_1a_3-a_2^2)}}, \quad m_1 = \sqrt{\frac{a_4(a_1a_3-a_2^2)}{\rho a_3^2}}, \quad m_2^2 = \frac{a_2^2 a_4}{\rho a_3^2}, \ p = s + m_1. \tag{30}$$

From (29) we get

$$k_1^2 - k_2^2 = -4m_0^2\sqrt{p^2+m_2^2}\cdot\sqrt{(p-2m_1)^2+m_2^2}. \tag{31}$$

Using (29)–(31), the expressions of the Laplace transforms of ths solutions can be written in equivalent forms, for which the inverse Laplace transforms can be obtained. We have:

$$\begin{aligned} u_1(x_1,t) = {} & \Phi_1(t)*\left[e^{-m_1t}J_0(m_2\sqrt{t^2+2m_0x_1t})\right]*\left[e^{m_1t}J_0(m_2\sqrt{t^2-2m_0x_1t}\right]e^{-2m_0m_1x_1}+ \\ & +\Phi_2(t)*\left[e^{-m_1t}H(t-m_0x_1)J_0(m_2\sqrt{t^2-m_0^2x_1^2})\right]*\left[e^{m_1t}H(t-m_0x_1)J_0(m_2\sqrt{t^2-m_0^2x_1^2})\right], \end{aligned} \tag{32}$$

where

$$\begin{aligned} \Phi_1(t) = {} & \frac{\alpha_1}{2m_0}\left[\mathrm{sh}(m_1(t-1))J_0(m_2(t-1)) - 2m_1\mathrm{ch}(m_1t)J_0(m_2t)+\right. \\ & \left.+(m_1^2+m_2^2)\int_0^t \mathrm{sh}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau\right] + \frac{\alpha_2}{8m_1m_0^3}\left[\mathrm{ch}(m_1t)J_0(m_2t)-\right. \\ & \left.-2m_1\int_0^t \mathrm{sh}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau + (\mathrm{m}_1^2+\mathrm{m}_2^2)\int_0^t (\mathrm{t}-\tau)\mathrm{ch}(\mathrm{m}_1\tau)\mathrm{J}_0(\mathrm{m}_2\tau)\mathrm{d}\tau\right], \end{aligned} \tag{33}$$

$$\begin{aligned} \Phi_2(t) = {} & -\frac{\alpha_1}{2m_0}\left[\mathrm{ch}(m_1(t-1))J_0(m_2(t-1)) - 2m_1\mathrm{sh}(m_1t)J_0(m_2t)+\right. \\ & \left.+(m_1^2+m_2^2)\int_0^t \mathrm{ch}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau\right] - \frac{\alpha_2}{8m_1m_0^3}\left[\mathrm{ch}(m_1\tau)J_0(m_2\tau)-\right. \\ & \left.-2m_1\int_0^t \mathrm{sh}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau + (\mathrm{m}_1^2+\mathrm{m}_2^2)\int_0^t (\mathrm{t}-\tau)\mathrm{ch}(\mathrm{m}_1\tau)\mathrm{J}_0(\mathrm{m}_2\tau)\mathrm{d}\tau\right], \end{aligned} \tag{34}$$

$$\alpha_1 = \frac{a_3 f_0 + a_2 b^0}{a_1 a_3 - a_2^2}, \quad \alpha_2 = \frac{a_4 f_0}{a_1 a_3 - a_2^2}, \tag{35}$$

$$\begin{aligned} P(x_1,t) = {} & \Big\{\Psi_1(t) * \left[e^{-m_1 t} J_0(m_2\sqrt{t^2+2m_0x_1t})\right] * \left[e^{m_1 t} J_0(m_2\sqrt{t^2-2m_0x_1t})\right] + \\ & + \frac{\beta_2}{8m_1m_0^3}\left[e^{-m_1(t-1)} J_0(m_2\sqrt{(t-1)^2+2m_0x_1(t-1)})\right] * \left[e^{m_1 t} J_0(m_2\sqrt{t^2-2m_0x_1t})\right]\Big\} e^{-2m_0m_1x_1} + \\ & + \Psi_2(t) * \left[e^{-m_1 t} H(t-m_0x_1) J_0(m_2\sqrt{t^2-m_0^2x_1^2})\right] * \left[e^{m_1 t} H(t-m_0x_1) J_0(m_2\sqrt{t^2-m_0^2x_1^2})\right] + \\ & + \frac{\beta_2}{8m_0^3}\left[e^{-m_1 t} H(t-m_0x_1) J_0(m_2\sqrt{t^2-m_0^2x_1^2}\right] * \left[e^{m_1 t} H(t-m_0x_1) J_0(m_2\sqrt{t^2-m_0^2x_1^2})\right], \end{aligned} \tag{36}$$

where

$$\begin{aligned} \Psi_1(t) = {} & \frac{\beta_1}{2m_0}\Bigg[\mathrm{sh}(m_1(t-1))J_0(m_2(t-1)) - 2m_1\mathrm{ch}(m_1t)J_0(m_2t) + \\ & + (m_1^2+m_2^2)\int_0^t \mathrm{sh}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau\Bigg] + \frac{\beta_2}{8m_1m_0^3}\cdot\frac{m_2}{t}\mathrm{ch}(m_1t)J_1(m_2t), \end{aligned} \tag{37}$$

$$\begin{aligned} \Psi_2(t) = {} & -\frac{\beta_1}{2m_0}\Bigg[\mathrm{ch}(m_1(t-1))J_0(m_2(t-1)) - 2m_1\mathrm{sh}(m_1t)J_0(m_2t) + \\ & + (m_1^2+m_2^2)\int_0^t \mathrm{ch}(m_1\tau)J_0(m_2\tau)\mathrm{d}\tau\Bigg] - \frac{\beta_2}{8m_1m_0^3}\cdot\frac{m_2}{t}\mathrm{sh}(m_1t)\cdot J_1(m_2t), \end{aligned} \tag{38}$$

and

$$\beta_1 = \frac{a_2 f_0 + a_1 b^0}{a_1 a_3 - a_2^2}, \quad \beta_2 = \frac{b^0 \rho}{a_1 a_3 - a_2^2}. \tag{39}$$

In the (32)–(39), J_0 and J_1 are the Bessel's functions and $u * v$ is the convolution of u and v defined by

$$(u * v)(x_1, t) = \int_0^t u(x_1, t-\tau)v(x_1,\tau)\mathrm{d}\tau. \tag{40}$$

From (10) and (36) we obtain the electric potential φ, and from (2) we find the stress tensor.

The term that contains the Heaviside's function $H(t - m_0x_1)$ has the character of an electroacoustic wave. The value $x_1 = \dfrac{t}{m_0}$ characterizes the wavefront.

Let be an arbitrary point of the considered medium, having the coordinates x_i^*. Before the moment $t^* = m_0 x_1^*$ the displacement have the expression

$$u_1(x_1^*,t) = \Phi_1(t) * \left[e^{-m_1 t} J_0(m_2\sqrt{t^2 + 2m_0 x_1^* t})\right] * \left[e^{m_1 t} J_0(m_2\sqrt{t_2 + 2m_0 x_1^* t})\right] e^{-2m_0 m_1 x_1^*}. \quad (41)$$

After the moment t^* we have

$$u_1(x_1^*,t) = \Phi_1(t) * \left[e^{-m_1 t} J_0(m_2\sqrt{t^2 + 2m_0 x_1^* t})\right] * \left[e^{m_1 t} J_0(m_2\sqrt{t^2 + 2m_0 x_1^* t})\right] e^{-2m_0 m_1 x_1^*} +$$
$$+\Phi_2(t) * \left[e^{-m_1 t} J_0(m_2\sqrt{t^2 - m_0^2(x_1^*)^2})\right] * \left[e^{m_1 t} J_0(m_2\sqrt{t^2 - m_0^2(x_1^*)^2})\right] \quad (42)$$

i.e. the displacement is discontinuous.

We obtain the analogous conclusion for the functions P, φ and the stress tensor.

The study of this problem, on the basis of Voigt's theory for a centrosymmetric izotropic piezolelectric medium shows that the wave propagation velocity is

$$v_1 = \sqrt{\frac{a_1}{\rho}}, \quad (43)$$

which is the same with that of a longitudinal wave in a pure elastic medium. This theory does not emphasizes the electromechanical coupling effect. In the case of Mindlin's model considered in this paper, the propagation velocity has the expression

$$v_2 = \frac{1}{2m_0} = \sqrt{\frac{a_1}{\rho} - \frac{a_2^2}{\rho a_3}}, \quad (44)$$

which clearly emphasizes the electromechanical coupling effect.

The decrease of the phase velocity of the longitudinal wave, therefore a "softening effect" appears for these materials.

References

1. A. Askar, P.C.Y. Lee, A.S. Cakmak, *The effect of surface curvature and discontinuity on the surface energy density and other induced fields in elastic dielectrics with polarization gradient.* Int. J. Solids Structures, **7**, pp. 523–537 (1971).
2. G.A. Maugin, *Continuum mechanics of electromagnetic solids.* North-Holland, Amsterdam, (1988).
3. K. Majorkowska-Knap, *Love's waves in elastic isotropic solid dielectrics.* Bull. Acad. Polon. Sci., Ser. Sci. Techn., **28**, pp. 615–622 (1980).
4. R.D. Mindlin, *Polarization gradient in elastic dielectrics.* Int. J. Solids, Structures, **4**, pp. 637–642 (1968).
5. G.E Roberts, H. Kaufman, *Table of Laplace Transforms.* W.B. Saunders Company, Philadelphia & London (1968).