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# SOME PROPERTIES OF THE COCYCLE GRAPH MATROIDS

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**Abstract.** Going on to a known Lemma, completely proved in a new way, and which gives us conditions, so that a set $X \subset E(G)$, of the graph edge-set, from $G = (V(G), E(G))$, connected or not connected, would be a cocycle in G, we introduce its links with the cocycle graph matroid $M^*(G)$. As a consequence of this Lemma, if $E(G)$ is a tree, or a forest in $G$, we find again, the situations when $B$ denotes a base in $M^*(G)$. We also prove the relation $M(G) = (M^*(G))^*$, between the cycle graph matroid $M(G)$ and the dual of the cocycle graph matroid $(M^*(G))^*$. Finaly we explain many other properties in $M^*(G)$, well maintained by suggestive examples.

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## 1 Preliminaries

We firstly introduce the cocycle graph notion, in conformity with [2], [3], [5], [6], [7].

**Definition 1.** *Let $G = (E(G), V(G))$ be a graph. A cocycle of G is a minimal edge-set $X \subset E(G)$, whose removal from G, increases the number of its connected components.*

**Remark 1.** *i)* In [5], for *cocycle*, Oxley uses *bond* and Recski, in [6] prefers *minimal cut-set* notion. At Berge, the *cocycle* is defined in [1], through an equivalent manner.

*ii)* A cocycle with an unique element, as $X = \{x\}$, is named *bridge*.

*Example 1.* The graph $G$, from Figure 1, has the cycle: $\{a, b, c\}$ and the cocycles: $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{d\}$, the last, being a bridge.

## 2 Results

Proposing it a new proof, we present the following:

**Lemma 1.** *Let $G = (V(G), E(G))$ be a graph. The subset $X \subset E(G)$ is cocycle of G, iff, its intersection with every maximal forest, for a graph G not connected, or with every maximal tree, for a graph G connected, is non-empty, X being minimal with this property.*

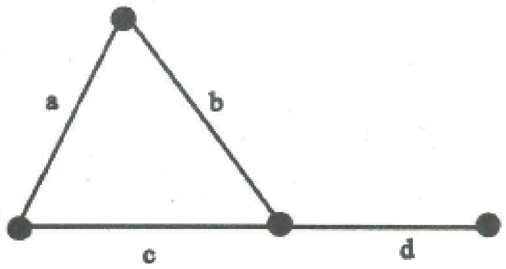

**Fig. 1.**

PROOF: When $G$ is not connected, the proof will be done by mathematical induction, about the number of its connected components. Without to diminish the generality, we shall suppose that $G$ is connected and we shall expose our proof, for maximal trees.

"$\Rightarrow$" We work in the hypothesis that $X$ is cocycle. So, $X$ is a minimal edge-set which disconnects the graph $G$. Let $T$ be a maximal (spanning) tree, namely, a connected graph. By reductio ad absurdum, we suppose that:

$$X \cap T = \emptyset. \tag{1}$$

Let $G \backslash X$ be the graph obtained from $G$, by removing the edges of $X$. From (1), we notice that $T$ contains all the edges of $G \backslash X$, and more, $T$ and $G \backslash X$ have the same vertex-set. So, $T$ and $G \backslash X$, have the same number of connected components, namely 1. On the other hand, $G \backslash X$ has no more connected components than $G$ has. Then, $G$ and $T$ have the same number of connected components, because $T$ is maximal tree. Therefor, we conclude that $T$, $G$, and $G \backslash X$ have the same number of connected components, and so, $X$ is not cocycle. The contradiction appeared considering (1), as possible.

"$\Leftarrow$" We work in the hypothesis:

$$T \cap X \neq \emptyset, \tag{2}$$

when $T$ is maximal tree for $G$ and $X$ is minimal, with the anterior property.

Also, by reductio ad absurdum, we suppose that $X$ is an edge-set, of $G$, which is not cocycle. Then, $T$, $G$ and $G \backslash X$ have the same number of connected components. Being contained in $G$, the maximal tree $T$ has the same vertex-set as $G$. More, $G$ and $G \backslash X$ compulsory have the same vertex-set. It occurs that $T$ and $G \backslash X$ have the same vertex-set. Since, for $T$ and $G \backslash X$ there are the same number of connected components, we deduce that $T$ is a maximal tree of $G \backslash X$, hence:

$$T \cap X = \emptyset.$$

This gives us a contradiction with (2)! Therefor, the supposition that $X$ is not cocycle, was eliminated, as false. ☐

Now, we can introduce a new notion in:

**Theorem 1.** *Let* $G = (V(G), E(G))$ *be a graph, with* $\mathcal{C}_*(G)$ *as cocycle set. Then:*

$$M^*(G) = (S = E(G), \mathcal{F}_*),$$

*which has* $\mathcal{C}_*(G)$ *as circuit set, is a new matroid, known as the* cocycle matroid *of* $G$. *Moreover, we have:*

$$M^*(G) = (M(G))^*, \tag{3}$$

*where* $(M(G))^*$ *is the dual of* $(M(G))$.

PROOF: We shall use the known **Theorem**, according to, $C^*$ is cocircuit in the matroid $M(G)$, iff, for any base $B \in \mathcal{B}$, it occurs that $C^* \cap B \neq \emptyset$, $C^*$ being minimal with this property (see [7]). Since, in the *cycle graph matroid* $M(G)$ (see [4]), the bases $B$ are forests, or maximal trees, in conformity with $G$, which is not connected or connected, through the above **Lemma 1**, we have that the cocircuits $C^*$, of $M(G)$, are cocycles in $G$. Because the cocircuit set $\mathcal{C}_*(G)$ defines the dual matroid $(M(G))^*$, it also follows that the cocycle set will generate the new matroid $M^*(G)$. More, the equality between $\mathcal{C}_*(G)$ and $\mathcal{C}^*(G)$ imposes:

$$M^*(G) = (M(G))^*,$$

namely, the relation (3). □

**Remark 2.** *i*) In the cocycle matroid $M^*(G)$, we will denote by $\mathcal{F}_*, \mathcal{B}_*, \mathcal{C}_*$ the independent, the base and respectively, the circuit set classes. The strict subsets, of the cocycles from $G$, are elements of $\mathcal{F}_*$.
*ii*) In conformity with a known **Theorem** (see [7], [8]) the cocycles of $G$ are the cocircuits for $M(G)$, so, they are circuits in $(M(G))^*$, or in $M^*(G)$.

*Example 2.* The cocycle graph matroid $M^*(G)$, from the Figure 1 is:

$$M^*(G) = (S = \{a, b, c, d\}, \mathcal{F}_* = \{\emptyset, \{a\}, \{b\}, \{c\}\}). \tag{4}$$

The cycle graph matroid is:

$$M(G) = (S = \{a, b, c, d\}, \mathcal{F}),$$

where :

$$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$$

and it has:

$$\mathcal{B} = \{\{a, b, d\}, \{a, c, d\}, \{b, c, d\}\},$$

as the base set.

Then, the bases of the dual matroid $(M(G))^*$ are:

$$\mathcal{B}^* = \{\{a\}, \{b\}, \{c\}\},$$

hence, we could write now, its form:

$$(M(G))^* = (S = \{a, b, c, d\}, \mathcal{F}^* = \{\emptyset, \{a\}, \{b\}, \{c\}\}). \tag{5}$$

From (4) and (5) we have the property (3) of **Theorem 1**.

**Corollary 1.** *Let $G = (V(G), E(G))$ be a graph and $M^*(G)$, its cocycle graph matroid. The set $B \subset E(G)$ is base in $M^*(G)$, iff, $E(G) \backslash B$ is a forest, or a maximal tree, according to $G$ which is not connected, respectively connected.*

PROOF: With **Theorem 1,** we find that $B$ is base in $M^*(G)$, iff, it is also a base in $(M(G))^*$. But, $B \in B^*$, iff, $S \backslash B = E(G) \backslash B \in B$, hence, iff the set $E(G) \backslash B$ is a forest, or a maximal tree, according to $G$ which is not connected, respectively connected. □

**Theorem 2.** *Let $G = (V(G), E(G))$ be a graph. Then, between the cycle graph matroid $M(G)$ and the cocycle graph matroid $M^*(G)$, the next relation follows:*

$$M(G) = (M^*(G))^* . \tag{6}$$

PROOF: We find (6), by passing to the dual, in the relation (3) of **Theorem 1,** or otherwise, if we rewrite it, in *co-notation.* Consequently we have:

$$(M^*(G))^* = \left((M(G))^*\right)^* , \tag{7}$$

and using the known property (see [8]):

$$\left((M(G))^*\right)^* = M(G), \tag{8}$$

from (7) and (8), it occurs (6). □

The last $M^*(G)$ property series, with an explicit new justification, will be presented in:

**Lemma 2.** *Let $G = (V(G), E(G))$ be a graph, and $M^*(G)$, its cocycle matroid. For $x \in E(G)$, we have the following equivalent sentences:*

*i) $\{x\}$ is a bridge in $G$;*
*ii) $\{x\}$ is circuit in $M^*(G)$;*
*iii) $\{x\}$ belongs none a base of $M^*(G)$;*
*iv) $\{x\}$ belongs to every base of $M(G)$.*

PROOF: *i)* $\Rightarrow$ *ii)*. This is obviously, with **Remark 2** *ii)*.
*ii)* $\Rightarrow$ *iii)*. By reductio ad absurdum, for $B_* \in \mathcal{B}_*$, we suppose that $\{x\} \subset B_*$. Because $\{x\}$ is dependent, it would mean that $B_*$ is dependent, which is an evident contradiction. Hence, $\{x\}$ could be in no base $B_* \in \mathcal{B}_*$.
*iii)* $\Rightarrow$ *iv)*. With **Theorem 1,** we deduce that a base $B_*$, of $M^*(G)$, coincides with a base $B^*$, of $(M(G))^*$, and more:

$$B_* = B^* = S \backslash B. \tag{9}$$

If $\{x\}$ is not in $B_* = B^* = S \backslash B$, it means that:

$$\{x\} \subseteq B \tag{10}$$

and (10) ends the proof of this implication.
*iv)* $\Rightarrow$ *i)*. If $\{x\}$ belongs to every base from $M(G)$, with the **Corollary 1,** it occurs that $\{x\}$ is in every forest, or maximal tree, according to $G$ which is not connected, respectively connected. Hence, its removing, from $G$, increases essentialy the number of its connected components. So, we deduce that $\{x\}$ is a bridge. □

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