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EUCLIDEAN MODULES

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Abstract. The main result of this paper is that a torsion-free cyclic module over a commutative ring with identity is an (so-called) Euclidean module if and only if the ring is an Euclidean ring.

1 Introduction

An Euclidean module is a natural extension of an Euclidean ring. Any nonzero submodule of an Euclidean R-module is a cyclic R-module. It is shown that a torsion-free cyclic R-module is Euclidean if and only if R is an Euclidean ring. The concept of side divisors and universal side divisors in a ring are generalized and studied in the cyclic modules. It is shown that a torsion-free cyclic R-module has no universal side divisors if and only if R has no universal side divisors. Also, a torsion-free cyclic R-module with no universal side divisors over an integral domain can never be an Euclidean R-module. Stable R-modules are defined and it is shown that any torsion-free cyclic R-module is stable if and only if R is a stable ring. Finally, it is shown that a stable torsion-free cyclic R-module over a principal ideal domain is an Euclidean R-module.

All rings (unless otherwise indicated) are commutative rings with identity and modules are unitary modules.

2 Main results

Definition. *Let N be the set of nonnegative integers and A a unitary R-module. A is an Euclidean R-module if there is a function $\phi : A \setminus \{0\} \to N$ such that (1) if $r \in R$, $a \in A$, $ra \neq 0$, and $a \neq 0$, then $\phi(a) \leq \phi(ra)$ and (2) if a and b are elements of A with $b \neq 0$, then there exist $r \in R$ and $c \in A$ such that $a = rb + c$ with $c = 0$ or $c \neq 0$ and $\phi(c) < \phi(b)$.*

Example 1. Every Euclidean ring R is an Euclidean R-module.

Example 2. Any nonzero submodule of an Euclidean R-module is an Euclidean R-module. Thus, every nonzero ideal of an Euclidean ring R is an Euclidean R-module.

Example 3. Let $R[X]$ be the Euclidean ring of polynomials over the field of real numbers with the Euclidean function $\phi(f) = \deg(f)$ for each nonzero element $f \in R[X]$. It is clear that the ideal $I = (X^2)$ is an Euclidean $R[X]$-module which is not an Euclidean ring under the function ϕ since $X^3 = XX^2 + 0$ and $X \notin I$.

Theorem 1. *Every nonzero submodule of an Euclidean R-module is a cyclic R-module.*

PROOF: If B is a nonzero submodule of an Euclidean R-module A with the Euclidean function ϕ, choose an element $b \in B$ such that $\phi(b)$ is the least integer in the set of nonnegative integers $\{\phi(x) \mid x \neq 0,\ x \in B\}$. If $a \in B$, then for some $r \in R$ and $c \in A$, $a = rb + c$ with $c = 0$ or $c \neq 0$ and $\phi(c) < \phi(b)$. Now, the minimality of $\phi(b)$ and the fact that $c \in B$ implies the desired result. □

Remark. It is not difficult to show that in any Euclidean R-module with the Euclidean function ϕ, $\phi(ra) = \phi(-ra)$ for all $r \in R$ and $a \in A$ whenever $ra \neq 0$ and $a \neq 0$. Moreover, for any generator a of A, $\phi(b) = \phi(a)$ if and only if b is a generator of A. Note that for any $c \in A$ and any generator a of A, we have always $\phi(a) \leq \phi(c)$. Recall that for any nonzero element r in an Euclidean ring R with the Euclidean function ϕ, $\phi(1_R) \leq \phi(r)$, and $\phi(r) = \phi(1_R)$ if and only if r is a unit in R.

Theorem 2. *A torsion-free cyclic R-module A is an Euclidean R-module if and only if R is an Euclidean ring.*

PROOF: For the sufficiency, let $A = \langle x \rangle$ be a torsion-free cyclic module over an Euclidean ring R with the Euclidean function ϕ. Define $\psi : A \setminus \{0\} \to N$ as $\psi(a) = \psi(rx) = \phi(r)$ for each nonzero a in A. The proof can be followed directly from the definition if we show that ψ is a well-defined function. Suppose, y is a generator of A. Thus, $x = ry = rsx$ implies $1_R = rs$ which makes $\psi(y) = \psi(sx) = \phi(s) = \phi(1_R) = \psi(x)$. Now, suppose $a = rx = sy$ for some nonzero r and s in R and y a generator of A. Hence, $a = rx = sy = stx$, for some $t \in R$, implies $r = st$ which makes $\psi(a) = \psi(rx) = \phi(r) \geq \phi(s) = \psi(sy)$. Again, by a similar argument, $\psi(sy) = \phi(s) \geq \phi(r) = \psi(rx)$. For the necessary part, suppose $A = \langle x \rangle$ and define $\psi : R \setminus \{0\} \to N$ as $\psi(r) = \phi(rx)$ for each nonzero r in R. Next, we show that ψ is a well-defined function on $R\setminus\{0\}$ and the rest of the proof which can be followed directly from the definition is left to the reader. Suppose $\psi(r) = \phi(rx)$ and $\psi(r) = \phi(ry)$ for an arbitrary generator y of A. Since A is an Euclidean R-module, $\phi(rx) = \phi(rsy) = \phi(sry) \geq \phi(ry)$ and similarly $\phi(ry) = \phi(rtx) = \phi(trx) \geq \phi(rx)$. Consequently, $\phi(rx) = \phi(ry)$. □

Definition. *For a cyclic R-module A, let $\tilde{A}$ denote the set of all generators of A together with 0. An element $a \in A \setminus \tilde{A}$ is said to be a side divisor of an element $b \in A$ whenever $(b - c)$ is a multiple of a for some $c \in \tilde{A}$. That is $(b - c) = ra$ for some $r \in R$. We shall call an element $a \in A \setminus \tilde{A}$ a universal side divisor in A whenever a is a side divisor of each element $b \in A$*

For a detailed study of "side divisors" and "universal side divisors" in a commutative ring, the reader is referred to [2].

Remark. It is not difficult to show that in a torsion-free cyclic R-module $A = \langle x \rangle$, rx is a generator of A if and only if r is a unit in R. Note that for the sufficient part, A needs not be a torsion-free R-module. Thus, $a = rx$ is in $A \setminus \tilde{A}$ if and only if r is in $R \setminus \tilde{R}$, or equivalently, $a = rx \in \tilde{A}$ if and only if $r \in \tilde{R}$ (where $\tilde{R}$ is the set of all units in R together with zero). Consequently, by virtue of the next theorem, $A = \langle x \rangle$ has no universal side divisors if and only if R has no universal side divisors.

Theorem 3. *For a cyclic R-module* $A =< x >$ *if* r *is a side divisor of* s *in* R, *then* rx *is a side divisor of* sx *in* A. *Conversely, for a cyclic torsion-free R-module* $A =< x >$ *if* $a = rx$ *is a side divisor of* $b = sx$ *in* A, *then* r *is a side divisor of* s *in* R.

PROOF: The proof can be followed directly from the definition and the first part of the above remark. □

Theorem 4. *A torsion-free cyclic R-module A over an integral domain can never be an Euclidean R-module if A has no universal side divisors.*

PROOF: In [3], it is shown that an integral domain with no universal side divisors cannot be an Euclidean domain. Now, the result is an immediate consequence of the above remark and Theorem 2. □

Definition. *An R-module A is said to be stable if whenever* $< a, b >= A$, *there exists* $r \in R$ *such that* $< a, b >=< a + rb >= A$.

See [2] for the definition of stable ring.

Theorem 5. *A torsion-free cyclic R-module A is stable R-module if and only if R is a stable ring.*

PROOF: We just give a proof for the necessary part and leave the other part to the reader. Suppose $A =< x >$ is a stable torsion-free R-module and $(r, s) = R$. Thus, $rr' + ss' = 1_R$ for some $r', s' \in R$. Consequently, $r'rx + s'sx = x$ implies $< rx, sx >= A$. Now, by hypothesis for some $t \in R$, $< rx, sx >=< rx + rsx >$ implies $rx + tsx = ux$ for some unit u in R. Hence, $(r + ts) = R$ which means that R is a stable ring. □

Theorem 6. *Any stable torsion-free cyclic R-module over a principal ideal domain R is an Euclidean R-module.*

PROOF: Since by Theorem 5.3 in [1] any stable principal ideal domain is an Euclidean domain, then the result is an immediate consequence of Theorem 5 and Theorem 22 above. □

Remark. In the following examples, it is shown that the converse of the above theorem is not always true.

Example 4. It is a well-known fact that Z, the ring of integers, is an Euclidean ring with the Euclidean function $\phi(n) = |n|$ for any nonzero $n \in Z$. Consequently, Z is an Euclidean Z-module which is not stable since $(3, 5) = Z$ is not stable in Z. Note tht 1 and -1 are the only units in Z. For a detailed study of stable rings and side divisors, see [2].

Example 5. In [2], it is shown that $R[X]$ is not a stable ring for any ring R. Thus, any nonzero ideal of $R[X]$ is an Euclidean $R[X]$-module which is not a stable $R[X]$-module whenever R is a field.

References

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