LIBERTAS MATHEMATICA, VOL XXII (2002)

MINIMAL POINT STATEMENTS IN PRODUCT STRUCTURES

MIHAI TURINICI

Abstract. Some technical enlargements of the minimal point statements in Goepfert, Tammer and Zălinescu [7, 8] are given. The basic tool for such an approach is a set of abstract ordering principles obtained under the ideas in Turinici [14].

AMS subject classification: Primary 46A40. Secondary 54E40.

Key words: Cone, order, complete metric space, bounded monotone sequence, minimal point, archimedean property, gauge function.

1 Introduction

Let (X, d) be a *complete metric space*; and Y, some (real) separated *locally convex space*. By a *convex cone* in Y we mean, as usually, any part L of Y with

(1D1) $L + L \subseteq L$; $\lambda L \subseteq L$, for all $\lambda > 0$; $0 \in L$.

In this case, the relation $\le$ (mod L) on Y defined as

(1D2) $y_1 \le y_2$ (mod L) if and only if $y_2 - y_1 \in L$

is reflexive and transitive; hence a *quasi-order*. Moreover, it is *compatible* with the linear structure of Y, in the sense

$$\begin{cases} y_1 \le y_2 \ (\mathrm{mod}\ L), \quad y \in Y,\ \lambda \ge 0 \Longrightarrow \\ y_1 + y \le y_2 + y \ (\mathrm{mod}\ L),\ \lambda y_1 \le \lambda y_2 \ (\mathrm{mod}\ L). \end{cases} \tag{1.1}$$

Assume further that $\{K, H\}$ is a pair of convex cones in Y with

(1H1) $K \subseteq H$ =closed (in the usual sense);

and pick some k^0 in K. We introduce a quasi-order $(\preceq) = (\preceq_K^{k^0})$ over $X \times Y$ by the convention

(1D3) $(x_1, y_1) \preceq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \le y_2 - y_1 (\mathrm{mod}\ K)$.

Finally, take some nonempty part A of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which the quasi-ordered structure $(A, \preceq)$ should have points with certain Zorn type *minimality* properties. A basic result in this direction obtained by Goepfert, Tammer and Zălinescu [7] in case

(1*H*2) $H = \mathrm{cl}(K)$ (where "cl"=the *closure* operator),

deals with convex cones K taken according to

(1*H*3) $K \setminus (-\mathrm{cl}(K)) \neq \emptyset$ [hence $K \neq \{0\}$],

and with elements $k^0 \in K \setminus (-\mathrm{cl}(K))$. The crucial assumption used by the quoted authors is

$$(1H4) \begin{cases} \text{if } ((x_n, y_n)) \subseteq A \text{ is } \preceq\text{-descending and } x_n \to x \text{ then } x \in P_X(A) \\ \text{and there exists } y \in A(x) \text{ such that } (x,y) \preceq (x_n, y_n), \text{ for all } n. \end{cases}$$

[Here, for each $(x,y) \in A$, $A(x)$ (resp., $A(y)$) stands for the x-section (resp., y-section) of (the relation) A; and P_X, P_Y are the *projection* operators from $X \times Y$ to X and Y respectively]. For the remaining ones, we need some preliminary facts. Let K^+ stand for the *dual* cone

(1*D*4) $K^+=\{y^* \in Y^*;\ y^*(y) \geq 0, \text{ for all } y \in K\}$,

where Y^* is the *topological dual* of Y. Denote also

(1*D*5) $K^+(k^0) = \{y^* \in K^+; y^*(k^0) = 1\}, \quad k^0 \in K \setminus \{0\}$.

Lemma 1.1. *The following equivalence is true*

$$k^0 \in K \setminus (-\mathrm{cl}(K)) \Longleftrightarrow K^+(k^0) \neq \emptyset. \tag{1.2}$$

PROOF: Let k^0 be arbitrary fixed in $K \setminus (-\mathrm{cl}(K))$. By a well known Hahn-Banach type result (see, for instance, Precupanu [12, ch. 5, Sect. 4]), k^0 and $-\mathrm{cl}(K) = \mathrm{cl}(-K)$ may be *strongly* separated by some (nonzero) element z^* of Y^*; that is

$$\alpha = z^*(k^0) > \beta \geq z^*(y), \text{ for all } y \in -\mathrm{cl}(K). \tag{1.3}$$

This immediately gives $\beta \geq 0$ (hence $\alpha > 0$) and $z^* \in K^+$ (by a standard reasoning); so, $y^* = \frac{1}{\alpha} z^*$ is an element of $K^+(k^0)$. Conversely, assume that $k^0 \in K \setminus \{0\}$ is taken according to $K^+(k^0) \neq \emptyset$; and let y^* be some point in $K^+(k^0)$. We have

$$y^*(y) \leq 0, \text{ for all } y \in -K;$$

wherefrom (by continuity)

$$y^*(y) \leq 0, \quad \text{for all} \quad y \in -\mathrm{cl}(K). \tag{1.4}$$

This shows that $k^0 \in -\mathrm{cl}(K)$ is impossible; and the proof is complete. □

So, without any loss, one may assume that the point $k^0 \in K \setminus (-\mathrm{cl}(K))$ is taken according to $K^+(k^0) \neq \emptyset$. The specific assumption to be used may be expressed in terms of a certain v^* of $K^+(k^0)$:

(1*H*5) $v^*(P_Y(A))$ ($\subseteq R$) is bounded below (in R).

Note that, a sufficient condition for this is

(1H6) $P_Y(A)$ is bounded below (mod K) $[\exists \tilde{y} \in Y\colon P_Y(A) \subseteq \tilde{y} + K]$.

The reciprocal is not in general true, as simple examples show. We are now in position to formulate the announced authors' contribution.

Theorem 1.1. *Let the general and specific hypotheses above be in use. Then, for each* $(x_0, y_0) \in A$ *there exists* $(\bar{x}, \bar{y}) \in A$ *in such a way that*

$$(\bar{x}, \bar{y}) \preceq (x_0, y_0); \tag{1.5}$$

and, moreover,

$$\textit{if } (x', y') \in A \ \textit{ fulfils } \ (x', y') \preceq (\bar{x}, \bar{y}) \ \textit{ then } \ x' = \bar{x},\ v^*(y') = v^*(\bar{y}). \tag{1.6}$$

This result extends some related statements in this arȩa due to Loridan [10]; and, as such, it includes the (classical by now) Ekeland's variational principle [6]. So, a technical development of its basic lines would be not without profit. In this direction, we note that Theorem 1.1 may be equally viewed as a *maximality* statement, with respect to the *dual* quasi-order $(\succeq) = (\succeq_K^{k^0})$ (over $X \times Y$):

(1D6) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_1 - y_2 (\text{mod } K)$.

Moreover, denote again by d the *semi-metric* (i.e.: non-sufficient metric) over $X \times Y$:

(1D7) $d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2), \quad (x_1, y_1), (x_2, y_2) \in X \times Y$;

and by Φ, the function (from $X \times Y$ to R):

(1D8) $\Phi(x, y) = v^*(y), \quad (x, y) \in X \times Y$.

The last conclusion of the result above may be then written as

$$(\bar{x}, \bar{y}) \succeq (x', y') \Longrightarrow d((\bar{x}, \bar{y}), (x', y')) = 0, \quad \Phi(\bar{x}, \bar{y}) = \Phi(x', y'). \tag{1.7}$$

This suggests us a possible deduction of Theorem 1.1 from a related ordering principle in Turinici [14]. (We refer to Section 2 for its exact formulation). It is our aim in the following to verify this assertion for a certain counterpart of Theorem 1.1 stated in the context below (cf. Section 4 for details):

(i) the regularity condition (1H2) is to be substituted by a weaker one, involving *archimedean closure*

(ii) the specific assumption (1H5) is expressed in terms of *conical gauge functions*.

All preliminary material for this is to be found in Section 3. And, in Section 5, the possibility of deriving genuine Zorn minimality principles from such results is analyzed. The obtained facts extend some contributions in this area due to Goepfert, Tammer and Zălinescu [8]. Some other aspects will be discussed elsewhere.

2 Abstract ordering principles

Let M be a nonempty set; and $\leq$ be a *quasi-order* over it. Further, let ρ be some *semi-metric* over M. The following ordering principle established in Turinici [14] is our starting point.

Theorem 2.1. *Suppose that*

$$(2H1)\ \begin{cases} (\rho,\leq) \text{ is normal [each } (\leq)\text{-ascending sequence in } M \text{ is a} \\ \rho\text{-Cauchy one, bounded from above].} \end{cases}$$

Then, for each $a_0 \in M$ there exists $\bar{a} \in M$ with

$$a_0 \leq \bar{a}; \tag{2.1}$$

and, moreover,

$$\text{if } a' \in M \text{ fulfils } \bar{a} \leq a', \text{ then } d(\bar{a}, a') = 0. \tag{2.2}$$

Note that, if the structure $(M, \leq, \rho)$ fulfils the extra assumption

(2H2) $a_1, a_2 \in M,\ a_1 \leq a_2,\ \rho(a_1, a_2) = 0 \Longrightarrow a_2 \leq a_1,$

the point $\bar{a}$ described by (2.2) is a *maximal* one (in the usual sense); and Theorem 2.1 becomes a variant of the well known Zorn maximality principle (cf. Bourbaki [2]). But, in the following, this will be not accepted. For a number of related aspects we refer to Altman [1].

A useful version of this result may be given under the lines below. Let $\varphi : M \to \bar{R} = R \cup \{-\infty, +\infty\}$ be a function with

(2H3) φ is $\leq$–decreasing ($a_1 \leq a_2 \Longrightarrow \varphi(a_1) \geq \varphi(a_2)$).

Theorem 2.2. *Suppose that $(\rho, \leq)$ is normal [in the sense of (2H1)]. Then, for each $a_0 \in M$ there exists $\bar{a} \in M$ fulfiling (2.1), as well as*

$$a' \in M,\ \bar{a} \leq a' \Longrightarrow d(\bar{a}, a') = 0,\quad \varphi(\bar{a}) = \varphi(a'). \tag{2.3}$$

PROOF: Without any loss, one may assume that (in addition to (2H3))

(2H4) φ is bounded in R ($-\infty < \inf \varphi(M) \leq \sup \varphi(M) < +\infty$).

For, otherwise, let χ be an *order isomorphism* between $\bar{R}$ and some *bounded* interval of R; such as, e.g.,

(2D1) $\chi(t) = \mathrm{arctg}(t),\ t \in R;\ \ \chi(-\infty) = -\pi/2,\ \chi(+\infty) = \pi/2.$

The composed function (from M to R)

(2D2) $\varphi_1(x) = \chi(\varphi(x)),\ \ x \in M$ (in short: $\varphi_1 = \chi \circ \varphi$)

fulfils (2H3) and (2H4). And, if the conclusion of Theorem 2.2 holds for φ_1 it will be also retainable for φ. Define another semi–metric $\sigma = \sigma_\varphi$ over M by the convention

(2D3) $\sigma(x,y) = \max\{\rho(x,y), |\varphi(x) - \varphi(y)|\}, \quad x,y \in M$.

Let (a_n) be some $(\leq)$–ascending sequence in M. By (2H1), (a_n) is a ρ–Cauchy sequence, bounded from above. On the other hand, (2H3)+(2H4) tell us that $(\varphi(a_n))$ is a descending and bounded sequence in R; hence a Cauchy one. Summing up, (a_n) is a σ–Cauchy sequence (bounded from above, as already said); and from this,

$$(\sigma, \leq) \text{ is normal (i.e., (2}H1\text{) holds).} \tag{2.4}$$

The conclusion to be derived is now a consequence of Theorem 2.1 applied to the structure $(M, \leq, \sigma)$. □

This result extends the one due to Brezis and Browder [3]. As far as we know, the idea of handling general (unbounded) functions goes back to Carja and Ursescu [4]. In general, Theorem 2.2 cannot be reduced to the Zorn maximality principle, unless our data are taken so as

(2H5) $a_1 \leq a_2, \rho(a_1,a_2) = 0, \varphi(a_1) = \varphi(a_2) \Longrightarrow a_2 \leq a_1$.

But, in what follows, conditions of this type are not accepted. So,we may ask of which is the relevance of this result in getting the quoted principle. As we shall see, a positive answer is available with respect to a certain *order* (i.e.: antisymmetric quasi–order) on M induced by our data. Precisely, denote by $\prec$ the relation (over M):

(2D4) $a_1 \prec a_2$ iff $a_1 \leq a_2$ and $\varphi(a_1) > \varphi(a_2)$.

(Note that, the alternative of $\prec$ having an empty graph in M^2 cannot be avoided, in general). The following facts are almost evident. (So, we omit the details).

Lemma 2.1. *The introduced relation is a strict order; i.e.,*

$$a \not\prec a, \quad \text{for each } a \in M \qquad \textit{(strict non-reflexive)} \tag{2.5}$$

$$a_1 \prec a_2,\ a_2 \prec a_3 \Longrightarrow a_1 \prec a_3 \qquad \textit{(transitive).} \tag{2.6}$$

As a consequence, the relation $(\preceq)$ over M, defined as

(2D5) $a_1 \preceq a_2$ iff either $a_1 \prec a_2$ or $a_1 = a_2$

is an *order* on M, which in addition is *coarser* than $(\leq)$:

$$a_1, a_2 \in M, \quad a_1 \preceq a_2 \Longrightarrow a_1 \leq a_2 \tag{2.7}$$

and fulfils the sufficiency property

$$a_1, a_2 \in M,\ a_1 \preceq a_2,\ \varphi(a_1) = \varphi(a_2) \Longrightarrow a_1 = a_2. \tag{2.8}$$

The usefulness of this construction is to be judged from

Theorem 2.3. *Let the conditions (2H1)+(2H3) be in force. Then, for each $a_0 \in M$, there exists $\bar{a} \in M$ with*

$$a_0 \preceq \bar{a} \tag{2.9}$$

and, moreover,

$$\text{if } a' \in M \text{ fulfils } \bar{a} \preceq a' \text{ then } \bar{a} = a'. \tag{2.10}$$

(In other words: $(\preceq)$ is a Zorn ordering).

Proof: We show that, in the precised setting, $(\rho, \preceq)$ is normal [i.e.: $(2H1)$ holds, with $(\preceq)$ in place of $(\leq)$]. In fact, let (a_n) be an $(\preceq)$–ascending sequence in M. By (2.7), this sequence is $(\leq)$–ascending; so, from $(2H1)$, (a_n) is ρ–Cauchy and bounded from above [modulo $(\leq)$]:

$$\exists\, a \in M: \ a_n \leq a_m \leq a, \quad \text{provided } n \leq m. \tag{2.11}$$

This, along with $(2H3)$, tells us that the (extended real) sequence $(\varphi(a_n))$ is descending and bounded from below:

$$\varphi(a_n) \geq \varphi(a_m) \geq \varphi(a), \text{ whenever } n \leq m. \tag{2.12}$$

If $(\varphi(a_n))$ is constant then, by (2.8), so is (a_n); and the conclusion is clear. Otherwise, we have relations like

$$\text{for each } n \text{ there exists } m > n \text{ with } \varphi(a_n) > \varphi(a_m) \text{ (hence } a_n \prec a_m). \tag{2.13}$$

But then (cf. Lemma 2.1 above)

$$\varphi(a_n) > \varphi(a) \text{ (hence } a_n \prec a), \quad \text{for each } n; \tag{2.14}$$

and the conclusion is again clear. On the other hand, (2.7) tells us that $(2H3)$ holds [modulo $(\preceq)$]. Summing up, Theorem 2.2 is applicable to $((M, \preceq, \rho); \varphi)$. Hence, for each $a_0 \in M$ there exists $\bar{a} \in M$ fulfiling the properties (2.1)+(2.3) [with $(\preceq)$ in place of $(\leq)$]. And this, along with (2.8), yields the conclusion we need. □

Remark. The core of our argument is the implication

$$(\rho, \leq) \text{ is normal } \Longrightarrow (\rho, \preceq) \text{ is normal.} \tag{2.15}$$

A natural question is of whether or not is this reversible. The answer is negative, in general. For, let the couple $((M, \leq, \rho); \varphi)$ be such that

$(2H6)$ $(\rho, \leq)$ is not normal (i.e., $(2H1)$ fails) and φ =constant.

The strict quasi-order $(\prec)$ attached to these data has an empty graph; so

$$a_1 \preceq a_2 \quad \text{if and only if } a_1 = a_2. \tag{2.16}$$

In other words, $(\rho, \preceq)$ is normal; but [cf. $(2H6)$] $(\rho, \leq)$ is not. Hence the claim.

3 Conical gauge functions

Let Y be a (real) vector space. Further, let L be some (non–degenerate) *convex cone* in Y. By a convention in Cristescu [5, ch. 5, Sect. 1], we say that L is *archimedean*, provided

(3D1) $k, y \in Y$ and $[\lambda k \leq y(\text{mod } L), \text{ for all } \lambda \geq 0] \Longrightarrow k \in -L$.

Assume in the following that

(3H1) L is an archimedean cone;

and also,

(3H2) $L \setminus (-L) \neq \emptyset$ (i.e.: L is not a linear subspace of Y).

Fix a certain $k^0 \in L \setminus (-L)$. Define the couple of multivalued functions (from Y to R)

(3D2) $\Gamma(y) = \{s \in R;\ k^0 s \leq y(\text{mod } L)\}$, $\Delta(y) = \{t \in R;\ y \leq k^0 t(\text{mod } L)\}$;

as well as their associated objects (from Y to $\bar{R}$)

(3D3) $\gamma(y) = \sup \Gamma(y)$, $\delta(y) = \inf \Delta(y)$, $y \in Y$.

(As usually, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = +\infty$). These will be referred to as the *gauge functions* attached to the cone L and the (nonzero) element k^0. It is our aim in what follows to study a few basic properties of the couple $\{\gamma, \delta\}$. Note that, in view of

$$\Gamma(y) = -\Delta(-y) \quad (\text{hence } \Delta(y) = -\Gamma(-y)), \tag{3.1}$$

one has

$$\gamma(y) = -\delta(-y) \quad (\text{hence } \delta(y) = -\gamma(-y)), \quad \text{for all such } y. \tag{3.2}$$

So, it will be suffice analyzing one of these (e.g., the latter) to derive the corresponding properties for its dual. However, for an easy reference, we shall write these properties for both functions of the couple.

(A) We start our developments by showing that

$$-\infty \notin \delta(Y) \quad (\text{resp., } +\infty \notin \gamma(Y)). \tag{3.3}$$

To verify this, note that for each $y \in Y$, the real subset $\Delta(y)$ (resp., $\Gamma(y)$) fulfils a hereditary property like

$$t \in \Delta(y),\ t' > t \Longrightarrow t' \in \Delta(y) \quad (s \in \Gamma(y),\ s' < s \Longrightarrow s' \in \Gamma(y)). \tag{3.4}$$

Now, assume by contradiction that

(3H3) $\delta(y_0) = -\infty$ (hence $\Delta(y_0) = R$), for some $y_0 \in Y$.

By the remark above, one has evaluations like

$$y_0 \leq k^0 t (\mathrm{mod}\ L),\ \forall t \in R;\ \text{hence } k^0 s \leq -y_0 (\mathrm{mod}\ L),\ \forall s \in R.$$

This, along with $(3H1)$, yields $k^0 \in -L$, contradiction; hence the claim. Note that, the alternative property

$$\text{(AP)} \qquad +\infty \in \delta(Y) \quad (\text{resp., } -\infty \in \gamma(Y))$$

cannot be avoided. So, we may ask of what can be said about the finite values of these functions. For a partial answer, note that

$$-\infty < \delta(y) < +\infty,\ \forall y \in k^0 R - L \quad (-\infty < \gamma(y) < +\infty,\ \forall y \in k^0 R + L). \tag{3.5}$$

Hence, in particular, one gets the useful fact

$$-\infty < \delta(y) \leq 0,\ \forall y \in -L \quad (0 \leq \gamma(y) < +\infty,\ \forall\ y \in L). \tag{3.6}$$

The global counterpart of it is to be given under the extra requirement

$(3H4)$ $\mathrm{aint}(L) \neq \emptyset$ (where "aint"=the *algebraic interior*).

Note that $(3H2)$ implies a regularity condition like

$(3H5)$ $0 \in Y$ is not an element of $\mathrm{aint}(L)$.

Conversely, this last requirement [and $(3H4)$] yields $(3H2)$; because, in such a case, $\mathrm{aint}(L) \subseteq L \setminus (-L)$. [The last assertion follows at once from the (set) relations

$$L + \mathrm{aint}(L) \subseteq \mathrm{aint}(L) \text{ (hence } L + \mathrm{aint}(L) = \mathrm{aint}(L)); \tag{3.7}$$

we do not give details]. Now, assume that the element k^0 is taken according to $k^0 \in \mathrm{aint}(L)$ (hence $k^0 \in L \setminus (-L)$). We claim that, necessarily,

$$+\infty \notin \delta(Y) \quad (\text{resp., } -\infty \notin \gamma(Y)). \tag{3.8}$$

In fact, let $y \in Y$ be arbitrary fixed. By the choice of k^0, there must be some $\varepsilon = \varepsilon(y) > 0$ in such a way that $k^0 + \lambda y \in L$, for each $\lambda \in [-\varepsilon, \varepsilon]$. In particular, when $\lambda = -\varepsilon$, this gives

$$y \in \frac{1}{\varepsilon} k^0 - L \text{ (wherefrom } \delta(y) \leq \frac{1}{\varepsilon});$$

and the assertion is proved.

(B) Return to the general setting of $(3H2)$. It is easy to see that the multifunctions $\{\Gamma, \Delta\}$ have the k^0–translation property

$$\Lambda(y + k^0 t) = \Lambda(y) + t, \quad \forall (y,t) \in Y \times R, \quad \forall \Lambda \in \{\Gamma, \Delta\}. \tag{3.9}$$

This yields a k^0–translation property for their associated functions $\{\gamma, \delta\}$:

$$\varphi(y + k^0 t) = \varphi(y) + t, \quad \forall (y,t) \in y \times R, \quad \forall \varphi \in \{\gamma . \delta\}. \tag{3.10}$$

In addition, by the very definition of these objects, one has (via ($3H2$))

$$\varphi(k^0 t) = t, \ \forall t \in R \text{ (hence, in particular, } \varphi(0) = 0); \tag{3.11}$$

where φ is as before. So, (combining with a previous conclusion) δ (resp., γ) is a *proper* function from Y to $R \cup \{+\infty\}$ (resp., $R \cup \{-\infty\}$).

(C) A useful property relating the couples (Γ, γ) and (Δ, δ) is

$$\begin{cases} \delta(y) \in \Delta(y), \ \text{whenever } \delta(y) < +\infty \\ (\gamma(y) \in \Gamma(y), \ \text{whenever } \gamma(y) > -\infty). \end{cases} \tag{3.12}$$

Indeed, by (3.4) above, one has

$$y - k^0\delta(y) \le k^0 s (\text{mod } L), \quad \text{for all } s > 0; \qquad \text{wherefrom}$$
$$t(y - k^0\delta(y)) \le k^0 (\text{mod } L), \ \text{for all } t \ge 0.$$

This, along with ($3H1$), establishes the assertion.

(D) We close these developments with the monotonicity properties for the couple (γ, δ). For example, one has

$$y_1 \le y_2 (\text{mod } L) \Longrightarrow \varphi(y_1) \le \varphi(y_2), \quad \forall \varphi \in \{\gamma, \delta\}. \tag{3.13}$$

This follows from the corresponding property related to (Γ, Δ):

$$y_1 \le y_2 (\text{mod } L) \Longrightarrow \Delta(y_1) \supseteq \Delta(y_2), \ \Gamma(y_1) \subseteq \Gamma(y_2). \tag{3.14}$$

Further aspects may be delineated under the regularity condition ($3H4$). Precisely, aint(L) is a convex cone without origin; that is, ($1D1$) holds without its last part. As a consequence, the object

($3D4$) $\text{Aint}(L) = \{0\} \cup \text{aint}(L)$

is a convex cone of Y, with the extra property (cf. (3.7))

$$\text{Aint}(L) \cap (-\text{Aint}(L)) = \{0\} \qquad (\textit{pointedness}). \tag{3.15}$$

Let $<$ (mod aint(L)) stand for the relation

($3D5$) $y_1 < y_2 (\text{mod aint}(L))$ if and only if $y_2 - y_1 \in \text{aint}(L)$.

This is a *strict order* (on Y) in the sense described by Lemma 2.1. Moreover, it is *compatible* with the linear structure of Y, in the sense

$$\begin{cases} y_1 < y_2 (\text{mod aint}(L)), & y \in Y, \ \lambda > 0 \Longrightarrow \\ y_1 + y < y_2 + y (\text{mod aint}(L)), & \lambda y_1 < \lambda y_2 (\text{mod aint}(L)). \end{cases} \tag{3.16}$$

Likewise, $\le$ (mod Aint(L)) is an order on Y, compatible with its linear structure (cf. (1.1)). In fact, it is nothing but the object attached to $<$ (mod aint(L)) under the model of ($2D5$); namely

$$y_1 \le y_2 (\text{mod Aint}(L)) \text{ iff either } y_1 < y_2 (\text{mod aint}(L)) \text{ or } y_1 = y_2. \tag{3.17}$$

The following statement is now available.

Lemma 3.1. *Let the precised conditions be in use. Then, δ (resp., γ) is strictly* $<$ (mod aint(L))*-increasing on $k^0R - L$ (resp., $k^0R + L$):*

$$\begin{cases} y_1, y_2 \in k^0R - L \text{ (resp. } k^0R + L), \ y_1 < y_2 (\text{mod aint}(L)) \\ \Longrightarrow \delta(y_1) < \delta(y_2) \text{ (resp., } \gamma(y_1) < \gamma(y_2)). \end{cases} \tag{3.18}$$

PROOF: By the very definition of the algebraic interior,

$$y_1 - y_2 \in -k^0\varepsilon - L, \quad \text{for some } \varepsilon > 0 \text{ (small enough).}$$

On the other hand, $\delta(y_2) < +\infty$ (cf. (3.5)), yields (via (3.12))

$$y_2 \in k^0\delta(y_2) - L;$$

so, by simply adding to the above

$$y_1 \in k^0(\delta(y_2) - \varepsilon) - L; \quad \text{hence } \delta(y_1) \leq \delta(y_2) - \varepsilon < \delta(y_2).$$

The proof is thereby complete. □

Remark. The finiteness condition involved in (3.18) cannot be removed. Indeed, let $y_1 \in Y$ be such that $\delta(y_1) = \infty$. If $y_2 \in Y$ fulfils $y_1 < y_2(\text{mod aint}(L))$ then, by (3.13), $\delta(y_1) \leq \delta(y_2)$; hence $\delta(y_2) = \infty$. This proves our claim.

Finally, by taking (3.6) into account, it follows that the restriction of δ (resp., γ) to $-L$ (resp., L) is strictly increasing:

$$\begin{cases} y_1, y_2 \in -L \text{ (resp, } L), \quad y_1 < y_2 (\text{mod aint}(L)) \\ \Longrightarrow \delta(y_1) < \delta(y_2) \quad (\text{resp., } \gamma(y_1) < \gamma(y_2)). \end{cases} \tag{3.19}$$

This may be useful in some concrete situations. Some related facts may be found in Goepfert, Tammer and Zălinescu [7].

4 Main results

We are now prepared to make precise the considerations developed in Section 1. To describe the working assumptions, we need some new concepts and auxiliary facts. Let Y be a (real) vector space. The notion of *archimedean* (convex) *cone* was already introduced in Section 3. Note that

$$\begin{cases} \text{the intersection of any (nonempty) family of} \\ \text{archimedean cones is an archimedean cone.} \end{cases} \tag{4.1}$$

So, for each (nonempty) part M of Y,

(4D1) $\text{arch}(M) = \cap \{L; M \subseteq L = \text{archimedean cone}\}$

is an archimedean cone including M, and minimal with these properties; we shall term it, the *archimedean closure* of M. Let (X, d) be a complete metric space; and $\{K, H\}$, a pair of convex cones in Y with

(4H1) $K \subseteq H =$ archimedean cone.

[For example, a good candidate for H is (cf. the above) $H = \mathrm{arch}(K)$. Moreover, if Y is taken as in Section 1, then (1H1) $\Longrightarrow$ (4H1); because any closed (convex) cone is archimedean]. Pick some $k^0 \in K$ and introduce the quasi-order $(\preceq) = (\preceq_K^{k^0})$ on $X \times Y$ by (1D3). Finally, take some nonempty part A of $X \times Y$. As in Section 1, we are interested to get sufficient conditions upon our data under which the quasi-ordered structure $(A, \preceq)$ should have points with certain Zorn type minimality properties. A basic answer to this problem is available for convex cones K taken according to

(4H2) $K \setminus (-H) \neq \emptyset$ (hence $K \neq \{0\}$),

and for elements $k^0 \in K \setminus (-H)$. [Note that, if Y is taken as in Section 1, then (1H3) is a particular case of this condition; because (cf. a previous remark), the choice (1H2) of H is allowed in the context of (4H1)]. The basic working hypothesis of these developments is again (1H4). And, the specific assumption to be used is formulated in terms of some $\varphi \in \{\gamma, \delta\}$, where

(4D2) $\gamma, \delta =$ the gauge functions attached to H and k^0.

Precisely, this may be written as

(4H3) $\varphi(P_Y(A))$ is a subset of R, bounded from below (in R).

The announced result may now be stated as

Theorem 4.1. *Let the precised conditions be in use. Then, for each $(x_0, y_0) \in A$, there exists $(\bar{x}, \bar{y}) \in A$ such that*

$$(\bar{x}, \bar{y}) \preceq (x_0, y_0); \tag{4.2}$$

and moreover

$$\text{if } (x', y') \in A \text{ fulfils } (x', y') \preceq (\bar{x}, \bar{y}) \text{ then } x' = \bar{x},\ \varphi(y') = \varphi(\bar{y}). \tag{4.3}$$

PROOF: Let $((x_n, y_n))$ be a $(\succeq)$-ascending (that is, $(\preceq)$-descending) sequence in A:

(4H4) $k^0 d(x_n, x_m) \leq y_n - y_m (\mathrm{mod}\ K)$, if $n \leq m$.

(Here, $(\succeq)$ is the *dual* of $(\preceq)$). By the choice (4H1) of H, one gets

$$k^0 d(x_n, x_m) \leq y_n - y_m (\mathrm{mod}\ H), \quad \text{whenever } n \leq m. \tag{4.4}$$

This, along with the finiteness k^0-translation and monotonicity properties of φ (cf. Section 3), yields

$$d(x_n, x_m) \leq \varphi(y_n) - \varphi(y_m), \quad \text{if } n \leq m. \tag{4.5}$$

The (real) sequence $(\varphi(y_n))$ is descending and (by (4H3)) bounded from below (in R); hence, a Cauchy sequence. This, added to (4.5), shows that (x_n) is d-Cauchy; and, as such, $x_n \to x$, for some $x \in X$. Combining with (1H4) yields $x \in P_X(A)$ and there exists an element $y \in A(x)$ with

$$(x_n, y_n) \succeq (x, y), \quad \text{for all } n. \tag{4.6}$$

In other words, $(d, \succeq)$ is normal over A (in the sense of (2H1)). Further, let the function $\Phi : X \times Y \to \bar{R}$ be introduced as

(4D3) $\Phi(x,y) = \varphi(y)$, $(x,y) \in X \times Y$ (i.e., $\Phi = \varphi \circ P_Y$).

Again by the monotonicity of φ, it follows that Φ is $\preceq$-increasing (or, equivalently, $\succeq$-decreasing) over A. Summing up, Theorem 2.2 is aplicable to the couple $((A, \succeq, d); \Phi)$. This firstly proves (4.2) (via (2.1)); and, secondly, (4.3) follows from (2.3). Hence the conclusion. □

It remains now to indicate concrete situations under which the regularity hypothesis (4H3) be fulfilled.

(*i*) A natural circumstance of this type is represented by the condition below (including (1H6)):

(4H5) $P_Y(A)$ is bounded below (mod H) $[\exists \widetilde{y} \in Y: \ P_Y(A) \subseteq \widetilde{y} + H]$.

Precisely, we claim that such a requirement is a particular case of (4H3) (the variant $\varphi = \gamma$). To verify this note that, without loss, one may express the written condition as

(4H6) $P_Y(A) \subseteq H$ (i.e.: $\widetilde{y} = 0$ in (4H5)).

[For, otherwise, passing to the subset $\widetilde{A}$ of $X \times Y$ defined as

(4D4) $(x,y) \in \widetilde{A}$ iff $(x, \widetilde{y} + y) \in A$,

this condition holds (in view of $P_Y(\widetilde{A}) = \widetilde{y} + P_Y(A)$) as well as (1H4). And,if the conclusion of Theorem 4.1 is retainable for $\widetilde{A}$, it will be also retainable for A]. But then, the (finite) positivity of γ over H yields the assertion. The corresponding version of our main result has a certain overlapping with Theorem 1.1 (when (4H5) is reduced to (1H6)). But, in general, these are distinct statements. Some related facts may be found in Tammer [13].

(*ii*) Another circumstance in this series refers to regularity conditions of the form below

(4H7) $\exists \widetilde{y} \in Y: \ P_Y(A) \subseteq \widetilde{y} + k^0 R - H, \ P_Y(A) \cap (\widetilde{y} - H) = \emptyset$.

Namely, we claim that such an assumption implies (4H3) (the variant $\varphi = \delta$). To do this note that, (by the same translation as above) one may assume $\widetilde{y} = 0$ in (4H7); i.e., this condition may be written as

(4H8) $P_Y(A) \subseteq k^0 R - H$ and $P_Y(A) \cap (-H) = \emptyset$.

The former of these tells us that δ is finite over $P_Y(A)$ (cf. Section 3). And, from the latter, one deduces

$$\delta(y) \geq 0, \quad \text{for all } y \in P_Y(A). \tag{4.7}$$

In fact, assume by contradiction that this would be not valid:

(4H9) $\delta(y_0) < 0$, for some $y_0 \in P_Y(A)$.

By the very definition of the left part, there exists some $\lambda_0 < 0$ with $y_0 \in k^0\lambda_0 - H$; wherefrom $y_0 \in -K - H \subseteq -H$, contradiction. This proves (4.7); and so, the claim follows. Note finally that the first part of (4H8) is clear, as soon as k^0 would be chosen according to

(4H10) $k^0 \in K \cap \operatorname{aint}(H)$ [hence $k^0 \in K \setminus (-H)$].

This follows at once from the developments in Section 3 (concerning the finiteness of δ). For a different approach of these problems we refer to Isac [9] and Nemeth [11].

5 Zorn minimal points

The results we just derived are *not genuine* Zorn minimality principles (as in Bourbaki [2]); because the quasi-orders appearing there are *not anti-symmetric* in general. So,it is natural to ask of whether or not is this removable. As we shall see below, such a device is possible, under the model of Theorem 2.3. Further aspects occasionated by these developments are also discussed.

Let the structures $\{X, Y\}$ be taken as in Section 4; and $\{K, H\}$, be a pair of (convex) cones in Y, fulfiling $(4H1)$. Further, pick some $k^0 \in K$; and construct the quasi-order $(\preceq) = (\preceq_K^{k^0})$ as in $(1D3)$. Given the (nonempty) part A of $X \times Y$, we may ask of which are the conditions upon our data so that coarser than $(\preceq)$ orders over A be available with the *standard* minimal Zorn property. As in Section 4, an appropriate answer is available under the regularity condition $(4H2)$. Namely, fix a certain $k^0 \in K \setminus (-H)$. The basic working hypotheses are $(1H4)$ and $(4H3)$; where, as precised in that place, φ is one of the *gauge* functions attached to H and k^0. Let also $\Phi : X \times Y \to \bar{R}$ stand for the function introduced as in $(4D3)$. The relation $(\sqsubset) = (\sqsubset_K^{k^0})$ over $X \times Y$ defined as

$(5D1)$ $(x_1, y_1) \sqsubset (x_2, y_2)$ iff $(x_1, y_1) \preceq (x_2, y_2)$ and $\Phi(x_1, y_1) < \Phi(x_2, y_2)$

is a *strict order* (cf. Lemma 2.1). Let $\sqsubseteq$ stand for its associated order (on $X \times Y$)

$(5D2)$ $(x_1, y_2) \sqsubseteq (x_2, y_2)$ if either $(x_1, y_1) \sqsubset (x_2, y_2)$ or $(x_1, y_1) = (x_2, y_2)$.

For the moment, $\sqsubseteq$ is *coarser* than $\preceq$ (over A), in the sense

$$(x_1, y_1), (x_2, y_2) \in A,\ (x_1, y_1) \sqsubseteq (x_2, y_2) \Longrightarrow (x_1, y_1) \preceq (x_2, y_2). \tag{5.1}$$

Concerning the converse inclusion, the following statement is true.

Lemma 5.1. *Assume that*

$(5H1)$ $y_1, y_2 \in P_Y(A),\ y_1 \leq y_2 (\text{mod } K),\ y_1 \neq y_2 \Longrightarrow \varphi(y_1) < \varphi(y_2)$.

Then, $\preceq$ is coarser than $\sqsubseteq$ over A; so, these relations are identical (over A).

PROOF: Let $(x_1, y_1), (x_2, y_2)$ be a couple of points in A with $(x_1, y_1) \preceq (x_2, y_2)$. We thus have (in particular) $y_1, y_2 \in P_Y(A)$ and $y_1 \leq y_2 (\text{mod } K)$. If $y_1 = y_2$, a relation like $d(x_1, x_2) \neq 0$ yields (by the choice of our data) $k^0 \in -H$, contradiction. So, necessarily, $d(x_1, x_2) = 0$; wherefrom $(x_1, y_1) = (x_2, y_2)$. If $y_1 \neq y_2$ one has, (by $(5H1)$) $\varphi(y_1) < \varphi(y_2)$ [hence $\Phi(x_1, y_1) < \Phi(x_2, y_2)$]. This, combined with our starting hypothesis, yields $(x_1, y_1) \sqsubset (x_2, y_2)$. The proof is complete. □

Let us now return to the initial framework (in which $(5H1)$ is excluded). The following Zorn (minimality) principle is available.

Theorem 5.1. *Let the precised conditions be admitted. Then, for each $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ such that*

$$(\bar{x}, \bar{y}) \sqsubseteq (x_0, y_0); \tag{5.2}$$

and, moreover,

$$\text{if } (x', y') \in A \text{ fulfils } (x', y') \sqsubseteq (\bar{x}, \bar{y}) \text{ then } (x', y') = (\bar{x}, \bar{y}). \tag{5.3}$$

(In other words: $\sqsubseteq$ is a Zorn ordering on A).

The proof is immediate, via Theorem 2.3, if we note that the *dual* strict order $\sqsupset$ and the *dual* order $\sqsupseteq$ are obtainable from the dual quasi-order $\succeq$ in the way described by $(2D4)+(2D5)$ (with Φ in place of φ). This result may be viewed as an algebraic version of the one due to Goepfert, Tammer and Zălinescu [7]. It tells us that coarser than $\preceq$ orders (on A) with a *standard* (minimal) Zorn property do exist. Moreover, if the (non-degenerate) convex cone K fulfils the regularity condition $(5H1)$, then (cf. Lemma 5.1), conclusions $(5.2)+(5.3)$ may be written with $\preceq$ in place of $\sqsubseteq$. An interesting circumstance of this type is to be described as follows. Assume that the couple of convex cones $\{K, H\}$ in Y (taken as before) fulfils the additional condition

$(5H2)$ $K \subseteq \mathrm{Aint}(H)[= \{0\} \cup \mathrm{aint}(H)]$.

Note that, in such a case (cf. the developments in Section 3)

$$K \text{ is pointed} \qquad (\text{because, so is } \mathrm{Aint}(H)). \tag{5.4}$$

As a consequence, the relation $(\preceq) = (\preceq_K^{k^0})$ given by $(1D3)$ is an order (on $X \times Y$). Let $\sqsubseteq$ stand for its associated order (on $X \times Y$) introduced as in $(5D2)$. A useful completion of Lemma 5.1 is now

Lemma 5.2. *Under the above conventions, the restrictions to A of $\preceq$ and $\sqsubseteq$ are identical; i.e., for $(x_1, y_1), (x_2, y_2) \in A$,*

$$(x_1, y_1) \preceq (x_2, y_2) \textit{ if and only if } (x_1, y_1) \sqsubseteq (x_2, y_2). \tag{5.5}$$

PROOF: It will suffice establishing that $(5H1)$ is fulfilled by our data. In fact, let $y_1, y_2 \in P_Y(A)$ be such that $y_1 \le y_2 (\mathrm{mod}\ K)$ and $y_1 \neq y_2$. By $(5H2)$, we have $y_1 < y_2 (\mathrm{mod}\ \mathrm{aint}(H))$; and this, combined with Lemma 3.1, yields $\varphi(y_1) < \varphi(y_2)$. Hence the conclusion. □

Now, by simply adding this to Theorem 5.1, one gets the following practical statement. (The general assumptions of this section prevail).

Theorem 5.2. *Under the precised setting, it is the case that: for each $(x_0, y_0) \in A$, there exists $(\bar{x}, \bar{y}) \in A$, in such a way that*

$$(\bar{x}, \bar{y}) \preceq (x_0, y_0); \tag{5.6}$$

and, moreover

$$\textit{if } (x', y') \in A \textit{ fulfils } (x', y') \preceq (\bar{x}, \bar{y}), \textit{ then } (x', y') = (\bar{x}, \bar{y}). \tag{5.7}$$

(In other words; $\preceq$ is a Zorn ordering on A).

This result extends a similar one due to Goepfert, Tammer and Zălinescu [8], provided Y is taken as in Section 1 and $(4H3)$ is to be used in the form described by $(4H7)$. Further aspects will be discussed elsewhere.

References

1. Altman M., *A generalization of the Brezis-Browder principle on ordered sets,* Nonlinear Analysis, 6, 157–165, (1982).
2. Bourbaki N., *Sur le theoreme de Zorn,* Archiv der Math., 2, 434–437, (1949/1950).
3. Brezis H. and Browder F.E., *A general principle on ordered sets in nonlinear functional analysis,* Adv.in Math., 21, 355–364, (1976).
4. Carja O. and Ursescu C., *The characteristics method for a first order partial differential equation,* An.Şt.Univ."Al.I.Cuza" Iaşi (S.I-a,Mat.), 39, 367–396, (1993).
5. Cristescu R., *Topological Vector Spaces,* Noordhoff Int. Publ., Leyden, 1977.
6. Ekeland I., *Nonconvex minimization problems,* Bull.Amer.Math.Soc. (New Series), 1, 443–474, (1979).
7. Goepfert A., Tammer C. and Zălinescu C., *On the vectorial Ekeland's variational principle and minimal points in product spaces,* Nonlinear Analysis, 39, 909–922, (2000).
8. Goepfert A., Tammer C. and Zălinescu C., *A new minimal point theorem in product spaces,* Zeitschr.Anal.Anwend. [J.Anal.Appl.], 18, 767–770, (1999).
9. Isac G., *The Ekeland's principle and Pareto ε-efficiency,* in "Multi–Objective Programming and Goal Programming (M.Tamiz ed.)", pp. 148–163, L.Notes in Econ.and Math.Systems vol.432, Springer, Berlin, 1996.
10. Loridan P., *ϵ-solutions in vector optimization problems,* J.Optim.Th. Appl., 43, 265–276, (1984).
11. Nemeth A., *A nonconvex vector minimization problem,* Nonlinear Analysis, 10, 669–678, (1986).
12. Precupanu T., *Linear Topological Spaces and Fundamentals of Convex Analysis (Romanian),* Ed.Acad. Române, Bucureşti, 1992.
13. Tammer C., *A generalization of Ekeland's variational principle,* Optimization, 25, 129–141, (1992).
14. Turinici M., *Vector extensions of the variational Ekeland's result,* An. Şt. Univ. "Al.I.Cuza" Iaşi (S.I-a, Mat.). 40, 225–266, (1994).