

A REFINEMENT OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS WHOSE DERIVATIVES BELONG TO L_∞ AND APPLICATIONS

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Abstract. A refinement of the Ostrowski inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications for special means, quadrature formulae, cumulative distribution functions and Jeffreys divergence measure are given.

1 Introduction

In 1938, A. Ostrowski [1], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b f(t) dt$ and the value $f(x)$, $x \in [a, b]$.

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \|f'\|_\infty (b-a), \quad (1.1)$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

In [4], S.S. Dragomir and S. Wang, by the use of the following integral identity,

$$f(x) - \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^b p(x, t) f'(t) dt, \quad x \in [a, b], \quad (1.2)$$

where $p : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x] \\ t - b & \text{if } t \in (x, b] \end{cases},$$

gave another proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating

the Riemann integral $\int_a^b f(t) dt$ by one arbitrary Riemann sum (see [4], Section 3). For related results, see the papers [2]-[16].

In this paper we point out a refinement of (1.1) and apply it for special means, in Numerical Analysis for quadrature formulae of Riemann type, for cumulative density functions in Probability Theory and for Jeffreys divergence measure in Information Theory.

2 Integral Inequalities

The following result, which is an improvement on Ostrowski's inequality, holds.

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) be an absolutely continuous function on $[a, b]$ whose derivative $f' \in L_\infty[a, b]$. Then*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{2.1} \\ & \leq \frac{1}{2(b-a)} \left[\|f'\|_{[a,x],\infty} (x-a)^2 + \|f'\|_{[x,b],\infty} (b-x)^2 \right] \\ & \leq \begin{cases} \|f'\|_{[a,b],\infty} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a); \\ \frac{1}{2} \left[\|f'\|_{[a,x],\infty}^p + \|f'\|_{[x,b],\infty}^p \right]^{\frac{1}{p}} \left[\left(\frac{x-a}{b-a} \right)^{2q} + \left(\frac{b-x}{b-a} \right)^{2q} \right]^{\frac{1}{q}} (b-a), \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty} \right] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^2 (b-a) \end{cases} \end{aligned}$$

for all $x \in [a, b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_\infty[m, n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty} = \operatorname{ess\,sup}_{t \in [m,n]} |g(t)| < \infty.$$

PROOF: Using the integration by parts formula for absolutely continuous functions on $[a, b]$, we have

$$\int_a^x (t-a) f'(t) dt = (x-a) f(x) - \int_a^x f(t) dt \tag{2.2}$$

and

$$\int_x^b (t-b) f'(t) dt = (b-x) f(x) - \int_x^b f(t) dt, \tag{2.3}$$

for all $x \in [a, b]$.

Adding these two equalities, we obtain the Montgomery identity (see for example [17, p. 565]):

$$(b-a) f(x) - \int_a^b f(t) dt = \int_a^x (t-a) f'(t) dt + \int_x^b (t-b) f'(t) dt \tag{2.4}$$

for all $x \in [a, b]$.

Taking the modulus, we deduce

$$\begin{aligned}
 & \left| (b-a)f(x) - \int_a^b f(t) dt \right| \\
 & \leq \left| \int_a^x (t-a)f'(t) dt \right| + \left| \int_x^b (t-b)f'(t) dt \right| \\
 & \leq \int_a^x (t-a)|f'(t)| dt + \int_x^b (b-t)|f'(t)| dt \quad (2.5) \\
 & \leq \|f'\|_{[a,x],\infty} \int_a^x (t-a) dt + \|f'\|_{[x,b],\infty} \int_x^b (b-t) dt \\
 & = \frac{1}{2} \left[\|f'\|_{[a,x],\infty} (x-a)^2 + \|f'\|_{[x,b],\infty} (b-x)^2 \right]
 \end{aligned}$$

and the first inequality in (2.1) is proved.

Now, let us observe that

$$\begin{aligned}
 & \|f'\|_{[a,x],\infty} (x-a)^2 + \|f'\|_{[x,b],\infty} (b-x)^2 \\
 & \leq \max \left\{ \|f'\|_{[a,x],\infty}, \|f'\|_{[x,b],\infty} \right\} \left[(x-a)^2 + (b-x)^2 \right] \\
 & = \max \left\{ \|f'\|_{[a,x],\infty}, \|f'\|_{[x,b],\infty} \right\} \left[\frac{1}{2} (b-a)^2 + 2 \left(x - \frac{a+b}{2} \right)^2 \right] \\
 & = (b-a)^2 \max \left\{ \|f'\|_{[a,x],\infty}, \|f'\|_{[x,b],\infty} \right\} \left[\frac{1}{2} + 2 \cdot \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] \\
 & = (b-a)^2 \|f'\|_{[a,b],\infty} \left[\frac{1}{2} + 2 \cdot \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right],
 \end{aligned}$$

and the first part of the second inequality in (2.1) is proved.

For the second inequality, we employ the elementary inequality for real numbers, which can be derived from Hölder's discrete inequality

$$0 \leq ms + nt \leq (m^p + n^p)^{\frac{1}{p}} \times (s^q + t^q)^{\frac{1}{q}}, \quad (2.6)$$

provided that $m, s, n, t \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Using (2.6), we obtain

$$\begin{aligned}
 & \|f'\|_{[a,x],\infty} (x-a)^2 + \|f'\|_{[x,b],\infty} (b-x)^2 \\
 & \leq \left(\|f'\|_{[a,x],\infty}^p + \|f'\|_{[x,b],\infty}^p \right)^{\frac{1}{p}} \left[(x-a)^{2q} + (b-x)^{2q} \right]^{\frac{1}{q}}
 \end{aligned}$$

and the second part of the second inequality in (2.1) is also obtained.

Finally, we observe that

$$\begin{aligned} & \|f'\|_{[a,x],\infty} (x-a)^2 + \|f'\|_{[x,b],\infty} (b-x)^2 \\ & \leq \max \left\{ (x-a)^2, (b-x)^2 \right\} \left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty} \right] \\ & = \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^2 \left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty} \right] \end{aligned}$$

and the last part of the second inequality in (2.1) is proved. \square

The following corollary is also natural.

Corollary 3. *Under the above assumptions, we have the mid point inequality*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{b-a}{8} \left[\|f'\|_{[a,\frac{a+b}{2}],\infty} + \|f'\|_{[\frac{a+b}{2},b],\infty} \right] \\ & \leq \begin{cases} \frac{1}{4} (b-a) \|f'\|_{[a,b],\infty}; \\ \frac{1}{2^{\frac{(3q-1)}{2}}} (b-a) \left[\|f'\|_{[a,\frac{a+b}{2}],\infty}^p + \|f'\|_{[\frac{a+b}{2},b],\infty}^p \right]^{\frac{1}{p}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned} \quad (2.7)$$

3 Applications to Special Means

Let us recall the following means of two positive numbers.

The Arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0.$$

The Geometric mean

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0.$$

The Harmonic mean

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0.$$

The Logarithmic mean

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0;$$

The Identric mean

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0;$$

The *p*-Logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b, \end{cases} \quad a, b > 0,$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

Denoting $L_0 := I$ and $L_{-1} := L$, then it is well known that L_p is monotonically increasing over $p \in \mathbb{R}$, and the following particular inequalities hold

$$H \leq G \leq L \leq I \leq A. \tag{3.1}$$

1. Consider the function $f : [a, b] \rightarrow \mathbb{R}$ ($0 < a < b < \infty$), $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1, 0\}$. Then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= L_p^p(a, b) = L_p^p, \\ \|f'\|_{[a,b],\infty} &= \gamma_p(a, b), \quad (\text{see also [2]}) \end{aligned}$$

where

$$\gamma_p(a, b) := \begin{cases} pb^{p-1} & \text{if } p \geq 1 \\ |p|a^{p-1} & \text{if } p \in (-\infty, 0) \cup (0, 1) \setminus \{-1\}. \end{cases}$$

Applying Theorem 2 for the function $f(x) = x^p$, we get

$$\begin{aligned} |x^p - L_p^p(a, b)| &\leq \frac{1}{2(b-a)} \left[\gamma_p(a, x)(x-a)^2 + \gamma_p(x, b)(b-x)^2 \right] \\ &\leq \begin{cases} \gamma_p(a, b) \left[\frac{1}{4} + \frac{(x-A(a,b))^2}{(b-a)^2} \right] (b-a) \\ \frac{1}{2} [\gamma_p^s(a, x) + \gamma_p^s(x, b)]^{\frac{1}{s}} \left[\left(\frac{x-a}{b-a} \right)^{2q} + \left(\frac{b-x}{b-a} \right)^{2q} \right]^{\frac{1}{q}} (b-a) \\ \text{where } s > 1, \frac{1}{s} + \frac{1}{q} = 1; \\ \frac{1}{2} [\gamma_p(a, x) + \gamma_p(x, b)] \left[\frac{1}{2} + \frac{|x-A(a,b)|}{b-a} \right]^2 (b-a) \end{cases} \tag{3.2} \end{aligned}$$

for all $x \in [a, b]$, which improves the inequality (2.1) from [2].

2. Consider the function $f : [a, b] \rightarrow \mathbb{R}$ ($0 < a < b < \infty$), $f(x) = \frac{1}{x}$. Then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= L_{-1}^{-1}(a, b), \\ \|f'\|_{[a,x],\infty} &= \frac{1}{a^2}, \quad \|f'\|_{[x,b],\infty} = \frac{1}{x^2} \end{aligned}$$

and then, from Theorem 2 applied for the function $f(x) = \frac{1}{x}$, we get

$$|x - L(a, b)| \leq \frac{1}{2(b-a)} \left[\left(\frac{x}{a} - 1 \right)^2 + \left(1 - \frac{b}{x} \right)^2 \right] xL(a, b)$$

$$\leq \begin{cases} \left[\frac{1}{4} + \frac{(x - A(a, b))^2}{(b - a)^2} \right] \frac{xL(a, b)}{a^2} (b - a) \\ \frac{1}{2} \left[\frac{a^{2s} + x^{2s}}{a^{2s}x^{2s}} \right]^{\frac{1}{2}} \left[\left(\frac{x - a}{b - a} \right)^{2q} + \left(\frac{b - x}{b - a} \right)^{2q} \right]^{\frac{1}{q}} xL(a, b) (b - a) \\ \text{where } s > 1, \frac{1}{s} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[\frac{a^2 + x^2}{a^2x^2} \right] \left[\frac{1}{2} + \frac{|x - A(a, b)|}{b - a} \right]^2 xL(a, b) (b - a) \end{cases} \quad (3.3)$$

for all $x \in [a, b]$, which improves the inequality (2.2) from [2].

3. Consider the function $f : [a, b] \rightarrow \mathbb{R}$ ($0 < a < b < \infty$), $f(x) = \ln x$. Then

$$\frac{1}{b - a} \int_a^b f(t) dt = \ln I(a, b),$$

$$\|f'\|_{[a, x], \infty} = \frac{1}{a}, \quad \|f'\|_{[x, b], \infty} = \frac{1}{x}$$

and then, by Theorem 2 applied for the function $f(x) = \ln x$, we obtain

$$\left| \ln \left[\frac{I(a, b)}{x} \right] \right| \leq \frac{1}{2(b - a)} \left[\frac{1}{a} (x - a)^2 + \frac{1}{x} (b - x)^2 \right]$$

$$\leq \begin{cases} \left[\frac{1}{4} + \frac{(x - A(a, b))^2}{(b - a)^2} \right] \frac{(b - a)}{a}; \\ \frac{1}{2} \left[\frac{a^s + x^s}{a^s x^s} \right]^{\frac{1}{2}} \left[\left(\frac{x - a}{b - a} \right)^{2q} + \left(\frac{b - x}{b - a} \right)^{2q} \right]^{\frac{1}{q}} (b - a) \\ \text{where } s > 1, \frac{1}{s} + \frac{1}{q} = 1; \\ \frac{1}{2} \left(\frac{a + x}{ax} \right) \left[\frac{1}{2} + \frac{|x - A(a, b)|}{b - a} \right]^2 (b - a) \end{cases} \quad (3.4)$$

for $x \in [a, b]$, which improves the inequality (2.3) from [2].

4 The Error Estimate in the Riemann Quadrature Formula

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a partitioning of the interval $[a, b]$ and define $h_i := x_{i+1} - x_i$, $\nu(h) = \max \{h_i | i = 0, \dots, n - 1\}$. Consider the following quadrature of Riemann type [2]

$$A_R(f, I_n, \xi) := \sum_{i=0}^{n-1} h_i f(\xi_i), \quad (4.1)$$

where $\xi = (\xi_0, \dots, \xi_{n-1})$ and $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n - 1$) are intermediate (arbitrarily chosen) points.

The following theorem holds.

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$ whose derivative $f' \in L_\infty [a, b]$. Then we have

$$\int_a^b f(t) dt = A_R(f, I_n, \xi) + R_R(f, I_n, \xi), \quad (4.2)$$

where $A_R(f, I_n, \xi)$ is the Riemann quadrature given by (4.1) and the remainder $R_R(f, I_n, \xi)$ in (4.2) satisfies the bound

$$|R_R(f, I_n, \xi)| \leq \frac{1}{2} \left[\sum_{i=0}^{n-1} \|f'\|_{[x_i, \xi_i], \infty} (\xi_i - x_i)^2 + \sum_{i=0}^{n-1} \|f'\|_{[\xi_i, x_{i+1}], \infty} (x_{i+1} - \xi_i)^2 \right]. \quad (4.3)$$

PROOF: Apply the first inequality in (2.1) on the interval $[x_i, x_{i+1}]$ to obtain

$$\left| h_i f(\xi_i) - \int_{x_i}^{x_{i+1}} f(t) dt \right| \leq \frac{1}{2} \left[\|f'\|_{[x_i, \xi_i], \infty} (\xi_i - x_i)^2 + \|f'\|_{[\xi_i, x_{i+1}], \infty} (x_{i+1} - \xi_i)^2 \right] \quad (4.4)$$

for all $i \in \{0, \dots, n-1\}$.

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we get the desired estimate (4.3). \square

Corollary 5. With the assumption of Theorem 4, we have the midpoint quadrature formula

$$\int_a^b f(t) dt = A_M(f, I_n) + R_M(f, I_n), \quad (4.5)$$

where $A_M(f, I_n)$ is the midpoint formula, i.e., we recall that

$$A_M(f, I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \quad (4.6)$$

and the remainder $R_M(f, I_n)$ satisfies the estimate

$$|R_M(f, I_n)| \leq \frac{1}{8} \sum_{i=0}^{n-1} \left[\|f'\|_{[x_i, \frac{x_i + x_{i+1}}{2}], \infty} + \|f'\|_{[\frac{x_i + x_{i+1}}{2}, x_{i+1}], \infty} \right] h_i^2. \quad (4.7)$$

Remark. a) As $\|f'\|_{[x_i, \xi_i], \infty}, \|f'\|_{[\xi_i, x_{i+1}], \infty} \leq \|f'\|_{[a, b], \infty}$, then

$$\begin{aligned} & \frac{1}{2} \left[\sum_{i=0}^{n-1} \|f'\|_{[x_i, \xi_i], \infty} (\xi_i - x_i)^2 + \sum_{i=0}^{n-1} \|f'\|_{[\xi_i, x_{i+1}], \infty} (x_{i+1} - \xi_i)^2 \right] \\ & \leq \|f'\|_{[a, b], \infty} \sum_{i=0}^{n-1} \left[\frac{(\xi_i - x_i)^2 + (x_{i+1} - \xi_i)^2}{2} \right] \\ & = \|f'\|_{[a, b], \infty} \sum_{i=0}^{n-1} \left[\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right], \end{aligned}$$

which proves that (4.3) is a better estimate than the one obtained in (2.1) from [2].

b) In addition, by the above considerations, we observe that

$$\begin{aligned} & \frac{1}{8} \sum_{i=0}^{n-1} \left[\|f'\|_{[x_i, \frac{x_i+x_{i+1}}{2}], \infty} + \|f'\|_{[\frac{x_i+x_{i+1}}{2}, x_{i+1}], \infty} \right] h_i^2 \\ & \leq \frac{1}{4} \|f'\|_{[a,b], \infty} \sum_{i=0}^{n-1} h_i^2, \end{aligned}$$

which shows that the estimate in (4.7) is better than the result (3.3) in [2].

c) Using the bounds provided by the second and third part of the second inequality in (2.1), the reader can obtain other bounds for the remainder $R_R(f, I_n, \xi)$ and $R_M(f, I_n)$. We omit the details.

5 Applications for Cumulative Density Functions

Let X be a random variable taking values in the finite interval $[a, b]$, with the cumulative distribution function $F(x) = \Pr(X \leq x)$ and the probability density function $f : [a, b] \rightarrow \mathbb{R}_+$.

The following theorem holds.

Theorem 6. Assume that $f : [a, b] \rightarrow \mathbb{R}$ belongs to $L_\infty[a, b]$. Then we have the inequality:

$$\begin{aligned} & \left| F(x) - \frac{b - E(X)}{b - a} \right| \\ & \leq \frac{1}{2(b-a)} \left[\|f\|_{[a,x], \infty} (x-a)^2 + \|f\|_{[x,b], \infty} (b-x)^2 \right] \\ & \leq \begin{cases} \|f\|_{[a,b], \infty} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \\ \left(\|f\|_{[a,x], \infty}^p + \|f\|_{[x,b], \infty}^p \right)^{\frac{1}{p}} \left[\left(\frac{x-a}{b-a} \right)^{2q} + \left(\frac{b-x}{b-a} \right)^{2q} \right]^{\frac{1}{q}} (b-a) \\ \text{where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left[\|f\|_{[a,x], \infty} + \|f\|_{[x,b], \infty} \right] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^2 (b-a) \end{cases} \quad (5.1) \end{aligned}$$

for all $x \in [a, b]$, where $E(X)$ is the expectation of X .

The proof follows by Theorem 2 applied for the cumulative function F and taking into account that

$$\int_a^b F(t) dt = F(t)t \Big|_a^b - \int_a^b tf(t) dt = b - E(X)$$

and

$$F'(t) = f(t), \quad t \in (a, b).$$

We now give an example for a Beta Random Variable.

We recall that a Beta Random Variable with parameters (p, q) has the probability density function

$$f(t; p, q) = \frac{t^{p-1} (1-t)^{q-1}}{B(p, q)}, \quad 0 < t < 1,$$

where

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt$$

is the Euler Beta function.

We observe that for $0 < p < 1$ (or $0 < q < 1$)

$$\|f(\cdot; p, q)\| = \sup_{t \in (0,1)} \left[\frac{t^{p-1} (1-t)^{q-1}}{B(p, q)} \right] = \infty.$$

Assume that $p, q \geq 1$. Then

$$\frac{df(t; p, q)}{dt} = \frac{t^{p-2} (1-t)^{q-2}}{B(p, q)} [-(p+q-2)t + q - 1].$$

We observe that

$$\frac{df(t; p, q)}{dt} = 0$$

iff $t_0 = \frac{p-1}{p+q-2} \in (0, 1)$ (for $p, q > 1$) and then $\frac{df(t; p, q)}{dt} > 0$ on $(0, t_0)$ and $\frac{df(t; p, q)}{dt} < 0$ on $(t_0, 1)$, i.e., the function $f(\cdot; p, q)$ is strictly increasing on $(0, t_0)$ and strictly decreasing on $(t_0, 1)$.

Consequently, we may deduce that

$$\begin{aligned} \|f(\cdot; p, q)\|_{[0,x],\infty} &= \begin{cases} \frac{f(x; p, q)}{B(p, q)} & \text{if } 0 < x < \frac{p-1}{p+q-2}, \\ \frac{f\left(\frac{p-1}{p+q-2}; p, q\right)}{B(p, q)} & \text{if } \frac{p-1}{p+q-2} \leq x < 1, \end{cases} \\ &= \begin{cases} \frac{x^{p-1} (1-x)^{q-1}}{B(p, q)} & \text{if } 0 < x < \frac{p-1}{p+q-2}, \\ \frac{(p-1)^{p-1} (q-1)^{q-1}}{B(p, q) (p+q-2)^{p+q-2}} & \text{if } \frac{p-1}{p+q-2} \leq x < 1, \end{cases} \end{aligned}$$

and, similarly,

$$\|f(\cdot; p, q)\|_{[x,1],\infty} = \begin{cases} \frac{(p-1)^{p-1} (q-1)^{q-1}}{B(p, q) (p+q-2)^{p+q-2}} & \text{if } 0 < x < \frac{p-1}{p+q-2}, \\ \frac{x^{p-1} (1-x)^{q-1}}{B(p, q)} & \text{if } \frac{p-1}{p+q-2} \leq x < 1. \end{cases}$$

Now, we may state the following proposition.

Proposition 7. Let X be a Beta random variable with the parameters (p, q) , $p, q \geq 1$. Then we have the inequality

$$\left| \Pr(X \leq x) - \frac{q}{p+q} \right| \leq \frac{1}{2B(p, q)} \times \begin{cases} x^{p+1} (1-x)^{q-1} + \frac{(1-x)^2 (p-1)^{p-1} (q-1)^{q-1}}{(p+q-2)^{p+q-2}} & \text{if } 0 < x < \frac{p-1}{p+q-2}, \\ \frac{(p-1)^{p-1} (q-1)^{q-1}}{(p+q-2)^{p+q-2}} x^2 + x^{p-1} (1-x)^{q+1} & \text{if } \frac{p-1}{p+q-2} \leq x < 1 \end{cases} \quad (5.2)$$

for all $x \in [0, 1]$.

The proof follows by (5.1), taking into account that for a Beta random variable

$$E(X) = \frac{p}{p+q}.$$

6 Applications for Jeffreys Distance in Information Theory

Assume that a set χ and the σ -finite measure μ are given. Consider the set of all probability densities on μ to be $\Omega := \{p|p: \chi \rightarrow \mathbb{R}, p(x) \geq 0, \int_{\chi} p(x) d\mu(x) = 1\}$. The Kullback-Leibler divergence [19] is well known among the information divergences. It is defined as:

$$D_{KL}(p, q) := \int_{\chi} p(x) \log \left[\frac{p(x)}{q(x)} \right] d\mu(x), \quad p, q \in \Omega, \quad (6.1)$$

where \log is to base 2.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v , *Hellinger distance* D_H [20], χ^2 -*divergence* D_{χ^2} , α -*divergence* D_{α} , *Bhattacharyya distance* D_B [21], *Harmonic distance* D_{H_a} , *Jeffreys distance* D_J [22], *triangular discrimination* D_{Δ} [23], etc. They are defined as follows:

$$D_v(p, q) := \int_{\chi} |p(x) - q(x)| d\mu(x), \quad p, q \in \Omega; \quad (6.2)$$

$$D_H(p, q) := \int_{\chi} \left[\sqrt{p(x)} - \sqrt{q(x)} \right]^2 d\mu(x), \quad p, q \in \Omega; \quad (6.3)$$

$$D_{\chi^2}(p, q) := \int_{\chi} p(x) \left[\left(\frac{q(x)}{p(x)} \right)^2 - 1 \right] d\mu(x), \quad p, q \in \Omega; \quad (6.4)$$

$$D_{\alpha}(p, q) := \frac{4}{1-\alpha^2} \left[1 - \int_{\chi} [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x) \right], \quad p, q \in \Omega; \quad (6.5)$$

$$D_B(p, q) := \int_{\chi} \sqrt{p(x)q(x)} d\mu(x), \quad p, q \in \Omega; \quad (6.6)$$

$$D_{H_a}(p, q) := \int_{\chi} \frac{2p(x)q(x)}{p(x)+q(x)} d\mu(x), \quad p, q \in \Omega; \quad (6.7)$$

$$D_J(p, q) := \int_{\chi} [p(x) - q(x)] \ln \left[\frac{p(x)}{q(x)} \right] d\mu(x), \quad p, q \in \Omega; \quad (6.8)$$

$$D_{\Delta}(p, q) := \int_{\chi} \frac{[p(x) - q(x)]^2}{p(x) + q(x)} d\mu(x), \quad p, q \in \Omega. \quad (6.9)$$

For other divergence measures, see the paper [24] by Kapur or the book on line [25] by Taneja. For a comprehensive collection of preprints available on line, see the RGMIA web site <http://rgmia.vu.edu.au/papersinfth.html>

The following inequalities involving the Jeffreys divergence are known (see for example the book on line by Taneja [25])

$$D_{H_a}(p, q) \geq \exp \left[-\frac{1}{2} D_J(p, q) \right], \quad p, q \in \Omega, \quad (6.10)$$

$$D_{H_a}(p, q) \geq 1 - \frac{1}{4} D_J(p, q), \quad p, q \in \Omega \quad (6.11)$$

and

$$D_J(p, q) \geq 4[1 - D_B(p, q)], \quad p, q \in \Omega, \quad (6.12)$$

where $D_{H_a}(\cdot, \cdot)$ is the Harmonic distance and $D_B(\cdot, \cdot)$ is the Bhattacharyya distance.

The following result holds (see also [26]).

Theorem 8. *We have the inequality*

$$2D_{\Delta}(p, q) \leq D_J(p, q) \leq \frac{1}{2} [D_{\chi^2}(p, q) + D_{\chi^2}(q, p)], \quad p, q \in \Omega, \quad (6.13)$$

where D_{χ^2} is the chi-square distance and D_{Δ} is the triangular discrimination.

PROOF: We use the celebrated Hermite-Hadamard inequality for convex functions

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2} \quad (6.14)$$

and choose $f(t) = \frac{1}{t}$ to get

$$\frac{2}{a+b} \leq \frac{\ln b - \ln a}{b-a} \leq \frac{a+b}{2ab},$$

which is equivalent to

$$\frac{2(b-a)^2}{a+b} \leq (b-a)(\ln b - \ln a) \leq \frac{a+b}{2ab}(b-a)^2. \quad (6.15)$$

If we choose in (6.15) $b = q(x)$, $a = p(x)$, $x \in \chi$, then we obtain

$$\begin{aligned} \frac{2(q(x) - p(x))^2}{p(x) + q(x)} &\leq (q(x) - p(x))(\ln q(x) - \ln p(x)) \\ &\leq \frac{p(x) + q(x)}{2p(x)q(x)}(q(x) - p(x))^2 \end{aligned}$$

and

$$\left(\frac{x}{a} - \frac{b}{x}\right)^2 > 0 \text{ for all } x \in [a, b] \text{ as}$$

$$\frac{x}{a} + \frac{b}{x} - 2 = 0$$

is equivalent to

$$x^2 - 2ax + ab = 0$$

and

$$\Delta = 4a^2 - 4ab = 4a(a - b) < 0 \quad (\text{as } 0 < a < b),$$

then we can conclude that a better inequality than (6.18) is that one for which $x = \sqrt{ab}$, getting:

$$0 \leq \frac{1}{\sqrt{ab}} - \frac{\ln b - \ln a}{b - a} \leq \frac{(\sqrt{b} - \sqrt{a})^2}{(b - a)a}, \quad a, b > 0, \quad (6.23)$$

which is equivalent with

$$0 \leq \frac{(b - a)^2}{\sqrt{ab}} - (b - a)(\ln b - \ln a) \leq \frac{|b - a|(\sqrt{b} - \sqrt{a})^2}{a} \quad (6.24)$$

for all $a, b \in (0, \infty)$.

Using the inequality (6.24) we may state the following theorem.

Theorem 12. For all $p, q \in \Omega$, we have

$$0 \leq \int_X \frac{(q(x) - p(x))^2}{\sqrt{p(x)q(x)}} d\mu(x) - D_J(p, q) \quad (6.25)$$

$$\leq \int_X \frac{|q(x) - p(x)| (\sqrt{q(x)} - \sqrt{p(x)})^2}{p(x)} d\mu(x),$$

provided that all the integrals exist.

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