LIBERTAS MATHEMATICA, VOL XXII (2002)

A QUADRATIC FREDHOLM INTEGRAL EQUATION AND ITS SOLUTION FOR VARIOUS KERNELS

MOSTAFA GHANDEHARI and MERLYND NESTELL

Abstract. Consider the Fredholm integral equation

$$\varphi(x) = 1 + \lambda\,\varphi(x)\int_0^1 k(x,y)\varphi(y)dy, \quad \lambda \text{ a real parameter.}$$

The solution of this equation is discussed for separable, difference and distribution kernels. Existence, uniqueness, and bifurcation questions are explored for various assumptions on the kernel.

1 Introduction

Consider the Fredholm quadratic integral equation

$$\varphi(x) = 1 + \lambda\,\varphi(x)\int_0^1 k(x,y)\varphi(y)dy, \tag{1}$$

where λ is a parameter. Equation (1) is a generalization of the Chandrasekhar H-equation

$$H(\mu) = 1 + \mu H(\mu)\int_0^1 \frac{\psi(\mu')}{\mu+\mu'}H(\mu')d\mu'. \tag{2}$$

Chandrasekhar used the H-function in the theory of radiative transfer [1].

We rewrite (1) as

$$\frac{\varphi(x)-1}{\varphi(x)} = \lambda\int_0^1 k(x,y)\varphi(y)dy \tag{3}$$

and substitute $\psi(x) = \dfrac{\varphi(x)-1}{\varphi(x)}$. Then (3) reduces to

$$\psi(x) = \lambda\int_0^1 k(x,y)\cdot\frac{1}{1-\psi(y)}\,dy. \tag{4}$$

Equation (4) is a Hammerstein equation that has the general form

$$\psi(x) + \int_0^1 k(x,y)f(y,\psi(y))dy = 0 \tag{5}$$

where ψ is the unknown and f is a nonlinear function. For a discussion of equations of Hammerstein type see Tricomi [7] or Corduneanu [3]. Dolph [4] has a treatment of nonlinear equations of Hammerstein type. The results of Dolph's paper are summarized by Corduneanu [3].

In contrast with linear theory, equation (1) is a quadratic Fredholm equation of the second kind. We also discuss the solution of a quadratic Fredholm equation of the first kind

$$g(x) = \lambda\varphi(x)\int_0^1 k(x,y)\varphi(y)dy \tag{6}$$

where g is a known function and φ is unknown. The solution of the integral equation (1) is treated under various assumptions on the kernel and a bifurcation analysis discussing existence and uniqueness for the parameter λ is presented in Section 2. In Section 3 solutions of (1) are presented with various assumptions on the kernel, and in Section 4 various generalizations of equation (1) are given. A knowledge of standard material on linear integral equations of Fredholm types with separable kernels is assumed as presented in Cochran [2].

2 Bifurcation

Consider the simple case of (1) where $k(x,y) \equiv 1$, that is, assume

$$\varphi(x) = 1 + \lambda\varphi(x)\int_0^1 \varphi(y)dy. \tag{7}$$

First, we note that solutions exist provided $\lambda \le \frac{1}{4}$. For example, if we look for a constant solution, then we obtain

$$\varphi = 1 + \lambda\varphi^2. \tag{8}$$

The discriminant of $\lambda\varphi^2 - \varphi + 1 = 0$ is equal to $1 - 4\lambda$. If we require $1 - 4\lambda \ge 0$ in order to obtain a real solution, then $\lambda \le \frac{1}{4}$. In that case, we obtain

$$\varphi = \frac{1 \pm \sqrt{1-4\lambda}}{2\lambda}. \tag{9}$$

From (8) we solve for λ to get

$$\lambda = \frac{\varphi - 1}{\varphi^2}. \tag{10}$$

Figure 1 gives a graph of λ in terms of φ. Note that the bifurcation curve can be interpreted as follows: There is no real solution of (8) for φ when $\lambda > \frac{1}{4}$, and when $\lambda = \frac{1}{4}$ there is one solution $\varphi = 2$. For $\lambda = 0$, we obtain the unique solution $\varphi = 1$. For $0 < \lambda < \frac{1}{4}$ and for $\lambda < 0$ there are two solutions. Tricomi [6] gives a bifurcation analysis of the quadratic integral equation

$$\varphi(x) - \lambda\int_0^1 \varphi^2(y)dy = 1, \tag{11}$$

where the bifurcation curve is the same as the one we have for (7). If we let $k(x, y) = \delta(y-x)$ in (1) we also obtain a quadratic equation where δ denotes the delta distribution. A real solution exists in this case if and only if $\lambda \leq \frac{1}{4}$.

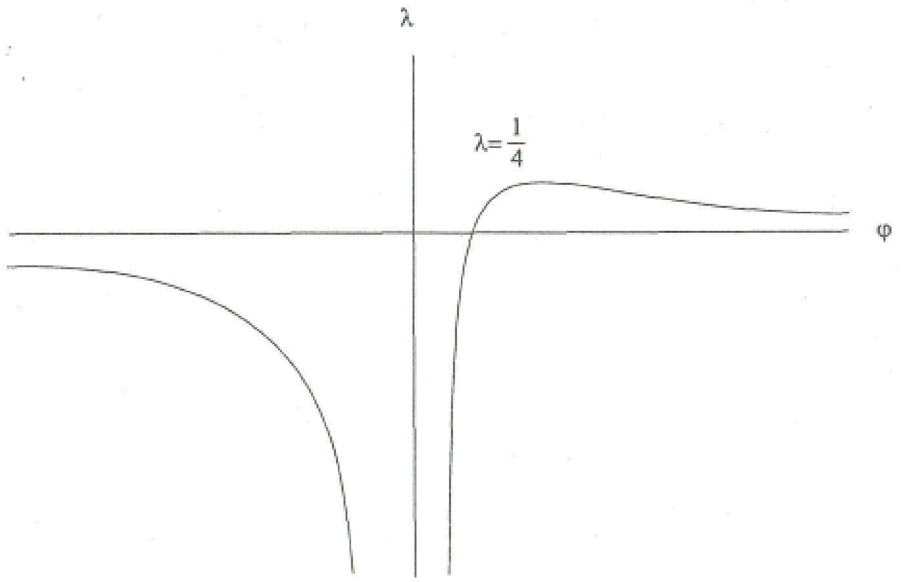

Fig. 1.

Consider a generalization of (7) in the form

$$\varphi(x) = 1 + \lambda\varphi^2(x) \int_0^1 \varphi(y)dy. \tag{12}$$

In order to look for a constant solution of (12) let $\varphi = K$ to obtain

$$K = 1 + \lambda K^3. \tag{13}$$

Then

$$\lambda = \frac{K-1}{K^3}. \tag{14}$$

The graph of λ in terms of K is given in Figure 2. At the critical value $\lambda = \frac{4}{27}$ there is a double solution for K. The discriminant of the cubic (13) yields the critical value $\lambda = \frac{4}{27}$. The results obtained by using Cardan's method of solution, Turnbull [7], agrees with the bifurcation curve given in Figure 2.

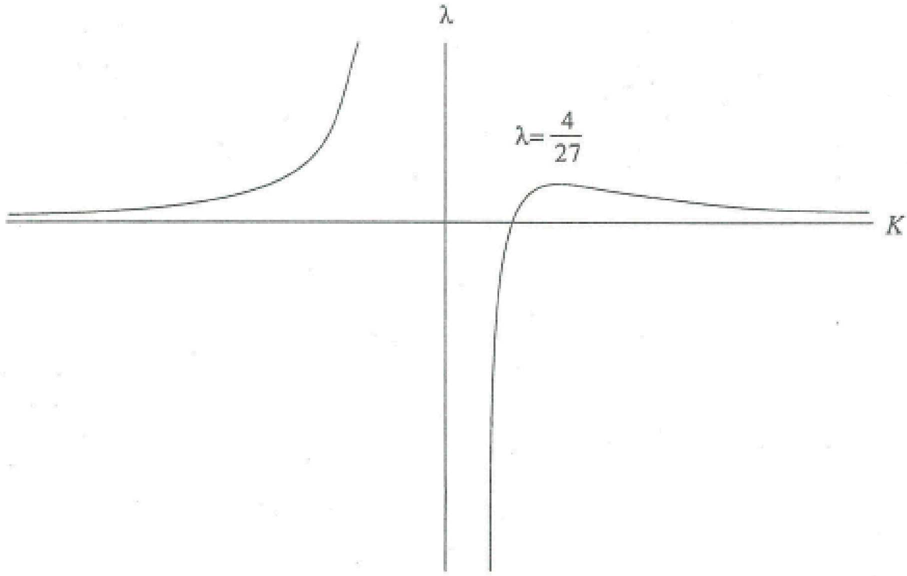

Fig. 2.

More generally, consider a nonlinear integral equation

$$\varphi(x) = 1 + \lambda\varphi^{n-1}(x)\int_0^1 \varphi(x)dx, \quad n \geq 2. \tag{15}$$

Then the constant solution satisfies

$$\varphi(x) = 1 + \lambda\varphi^n(x). \tag{16}$$

Analyzing the bifurcation curve for (15) we have a behavior similar to that depicted in Figure 1 for n even and similar to that in Figure 2 for n odd. If we solve for λ in (16) we obtain

$$\lambda = \frac{\varphi - 1}{\varphi^n}. \tag{17}$$

The critical value of $\lambda = \dfrac{(n-1)^{n-1}}{n^n}$ behaves as follows: For n even, if $\lambda \leq \dfrac{(n-1)^{n-1}}{n^n}$, there is a solution of (16). For $0 < \lambda \leq \dfrac{(n-1)^{n-1}}{n^n}$, there are two solutions. For $\lambda = 0$, there is the unique solution $\varphi = 1$. For $\lambda < 0$, there are two solutions. If n is odd, there is a unique solution for $\lambda \geq \dfrac{(n-1)^{n-1}}{n^n}$. There are three solutions for $0 < \lambda < \dfrac{(n-1)^{n-1}}{n^n}$, and a unique solution for $\lambda \leq 0$.

In passing, it is interesting to consider the following result regarding the complex case. Assume $\varphi = \varphi_1 + i\varphi_2$ where φ_1 and φ_2 are the real and imaginary parts of φ. Substituting

φ in (7) and equating real and imaginary parts, we obtain

$$\varphi_1 = 1 + \lambda \left[\varphi_1 \int_0^1 \varphi_1 - \varphi_2 \int_0^1 \varphi_2 \right] \tag{18}$$

$$\varphi_2 = \lambda \left[\varphi_2 \int_0^1 \varphi_1 + \varphi_1 \int_0^1 \varphi_2 \right] \tag{19}$$

From (18) and (19) we find

$$\varphi_1^2 + \varphi_2^2 = \frac{1}{\lambda} \frac{\varphi_2}{\int_0^1 \varphi_2}. \tag{20}$$

If we let $\alpha = \frac{1}{\lambda} \frac{1}{\int_0^1 \varphi_2}$, then we obtain the circle $\varphi_1^2 + \left(\varphi_2 - \frac{\alpha}{2}\right)^2 = \frac{\alpha^2}{4}$ in the (φ_1, φ_2) plane.

In the simple case of a constant solution given by (9), we can directly check that $\varphi_1^2 + \varphi_2^2 = \frac{1}{\lambda}$. Thus, for complex solutions the real and imaginary parts of φ are located on a circle.

3 Separable Kernels.

Next, we discuss the solution of (1) for a separable kernel

$$k(x,y) = \sum_{i=1}^{n} A_i(x) B_i(y). \tag{21}$$

First consider a special case for $n = 1$.

$$k(x,y) = A(x)B(y). \tag{22}$$

Substitute (22) in (1) to obtain

$$\varphi(x) = 1 + \varphi(x) \int_0^1 A(x)B(y)\varphi(y)dy. \tag{23}$$

Assume

$$\alpha = \int_0^1 B(y)\varphi(y)dy. \tag{24}$$

Substitute (24) in (23), and solve for $\varphi(x)$ to obtain

$$\varphi(x) = \frac{1}{1 - \lambda\,\alpha\,A(x)}. \tag{25}$$

Substituting (25) in (24), we obtain

$$\alpha = \langle \varphi, B \rangle = \left\langle \frac{1}{1 - \lambda \alpha A}, B \right\rangle.$$

That is, we get

$$\alpha = \int_0^1 \frac{B(x)}{1 - \lambda\,\alpha\,A(x)} dx. \tag{26}$$

In contrast with the linear theory, equation (26) is a nonlinear equation in α. Even in a simple case we obtain a transcendental equation in α. For example, if $B(x) = 1$, $A(x) = x$, we can integrate to obtain $\lambda\alpha^2 + \ln(1 - \lambda\alpha) = 0$. If we assume $A(x) = \sin x$, $B(x) = \cos x$, then we also obtain $\lambda\alpha^2 + \ln(1 - \lambda\alpha) = 0$.

Next we consider another special case of (21) with $n = 2$. If we let $k(x, y) = \sin(x - y)$, then in (27) we let $A_1(x) = \sin x$, $A_2(x) = \cos x$, $B(y) = \cos y$, $B_2(y) = -\sin y$. Assume

$$\alpha_1 = \langle B_1, \varphi\rangle \quad \text{and} \quad \alpha_2 = \langle B_2, \varphi\rangle. \tag{27}$$

Let the limits of integration in (1) be 0 and $\frac{\pi}{2}$. Using (27) in (1) we find that φ has the form

$$\varphi(x) = \frac{1}{1 - \lambda \sum_{i=1}^{2} \alpha_i A_i(x)}. \tag{28}$$

Substituting (28) in (27) we obtain

$$\alpha_1 = \int_0^1 \frac{B_i(x)}{1 - \lambda \sum_{i=1}^{n} \alpha_i A_i(x)} dx, \quad i = 1, 2. \tag{29}$$

In the case that $k(x, y) = \sin(x - y)$, we obtain

$$\alpha_1^2 + \alpha_2^2 = \ln\left(\frac{1 - \lambda\alpha_2}{1 - \lambda\alpha_1}\right). \tag{30}$$

The latter equation can be used with (29) to approximate α_1 and α_2 Saaty [5].
Assume $A_i(x) = 1$, $i = 1, \cdots, n$. Let $\beta_i = \int_0^1 B_i(y)dy$. Then

$$\alpha_i = \frac{1}{1 - \lambda \sum \alpha_i} \beta_i. \tag{31}$$

Then

$$\left(\sum_{i=1}^{n} \alpha_i\right)\left(1 - \lambda \sum_{i=1}^{n} \alpha_i\right) = \sum_{i=1}^{n} \beta_i. \tag{32}$$

We can use the quadratic equation to solve for $\sum_{i=1}^{n} \alpha_i$ and then use (31) to obtain α_i. Note that (32) has a solution if and only if $\lambda \leq \frac{1}{4}$.

In general, we obtain a system of nonlinear equations by sustituting (21) in (1) to obtain

$$\begin{aligned} \varphi(x) &= 1 + \lambda\varphi(x) \int_0^1 \sum_{i=1}^{n} A_i(x) B_i(y) \varphi(y) dy = \\ &= 1 + \lambda\varphi(x) \sum_{i=1}^{n} A_i(x) \int_0^1 B_i(y)\varphi(y) dy. \end{aligned} \tag{33}$$

Define an inner product

$$\alpha_i = \int_0^1 B_i(y)\varphi(y)dy = \langle B_i, \varphi \rangle. \tag{34}$$

Using (34) in (33) we find that φ has the form

$$\varphi(x) = \frac{1}{1 - \lambda \sum_{i=1}^{n} \alpha_i A_i(x)}. \tag{35}$$

Substituting (35) in (34) we obtain

$$\alpha_i = \langle \varphi, B_i \rangle = \left\langle \frac{1}{1 - \lambda \sum\limits_{i=1}^{n} \alpha_i A_i}, B_i \right\rangle, \tag{36}$$

that is, we get

$$\alpha_i = \int_0^1 \frac{B_i(x)}{1 - \lambda \sum\limits_{i=1}^{n} \alpha_i A_i(x)} dx, \quad i = 1, \ldots, n. \tag{37}$$

In contrast with the linear theory, the system (37) is a nonlinear system of equations in terms of α_i, $i = 1, \ldots, n$.

4 Existence and Uniqueness of Solutions

Now we discuss (1) by letting $\psi(x) = \dfrac{\varphi(x) - 1}{\varphi(x)}$, that reduces it to an equation of Hammerstein type. Tricomi [6] uses successive approximations to give conditions under which equations of Hammerstein type have solutions. We use equation (4) and a geometric series to give a lower bound for λ.

Assume $|\psi(y)| < 1$, so that we can use a geometric series on the right hand side of (4). Taking the absolute value of both sides we obtain

$$\begin{aligned} |\psi(x)| &= |\lambda| \left| \int_0^1 (k(x,y) \sum_{i=0}^{\infty} \psi^i(y)) dy \right| \\ &\leq |\lambda| \sum_{i=0}^{\infty} \int_0^1 |k(x,y)| |\psi^i(y)| dy \end{aligned} \tag{38}$$

Let $K(x) = ||k(x,y)||$ with respect to y. Then, by the Cauchy-Schwartz inequality,

$$|\psi(x)| \leq \lambda \sum_{i=0}^{\infty} K(x) \, ||\psi||^i, \tag{39}$$

$K(x) = ||k(x,y)||$ with respect to y. Since $|\psi| < 1$, we obtain $||\psi|| < 1$. Then, we use a geometric series to obtain

$$|\psi(x)| \leq |\lambda| \frac{1}{1 - ||\psi||}.$$

Thus, we conclude

$$\|\psi\| \leq |\lambda| \frac{1}{1 - \|\psi\|} K, \tag{40}$$

where $\|K(x)\| \leq K$. Let $\beta = \|\psi\| < 1$, then from (33) we obtain the quadratic inequality

$$\beta^2 - \beta + |\lambda| K \geq 0. \tag{41}$$

In order for $\beta^2 - \beta + |\lambda| K \geq 0$, we obtain

$$1 - 4|\lambda| K \leq 0. \tag{42}$$

Thus, there is a solution ψ with the assumptions mentioned if

$$|\lambda| \geq \frac{1}{4K}. \tag{43}$$

Our work could be extended by examining different kinds of kernels from the ones we have investigated.

Acknowledgement

The authors are thankful to Professor Rangachary Kannan and P.M. Anselone for useful discussions related to this article.

References

1. S. Chandrasekhar, *Radiative Transfer*, Oxford, Clarendon Press, 1950.
2. J. A. Cochran, *Applied Mathematics, Principles, Techniques, and Applications*, Wadsworth Inc., 1982.
3. C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.
4. C.L. Dolph, *Nonlinear Integral Equations of Hammerstein type*, Trans. Am. Math. Soc. 66 (1949), 289–307.
5. T.L. Saaty, *Modern Nonlinear Equations*, McGraw-Hill, 1967.
6. F.G. Tricomi, *Integral Equations*, Interscience Publishers, 1957.
7. H.W. Turnbull, *Theory of Equations*, Oliver and Boyd Publishers, 1939.