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ON A GENERALIZED HEMIVARIATIONAL INEQUALITY ON REFLEXIVE BANACH SPACES

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Abstract. We prove two existence theorems for the case of Hemivariational Inequalities on reflexive Banach spaces, using minimax results and recession analysis concepts. Several applications are provided in the final of the paper.

1 Introduction

The mathematical theory of hemivariational inequalities and their applications in mechanics, engineering or economics, were introduced and developed by P.D. Panagiotopoulos ([20], [21], [22], [23], [24], [26]). This theory has been developed in order to fill the gap existing in the variational formulations of boundary value problems (B.V.P.s) when nonsmooth and generally nonconvex energy functions are involved in the formulations of the problem. In fact, this theory of hemivariational inequalities may be considered as an extension of the theory of variational inequalities ([8], [10], [15], [13]). For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs ([21], [26], [19], [18]).

The purpose of this paper is to extend these results in the framework of hemivariational inequalities governed by two variable operators. We shall study a generalized hemivariational inequality and its applications to hemivariational-variational problems, as well as in engineering. To be more specific, we establish several existence theorems of generalized hemivariational inequalities for two-variable set-valued mappings. The proofs of these results appeal to Ky Fan's Inequality and to reccesion analysis concepts. As applications, we first derive alternative results which concerns existence theorems of Hartman-Stampacchia type (see [27] for more detail), and then the existence theorems of solutions for the variational inequalities given by a semi-monotone operator with two variables (see [5], [11] for comprehensive material). Recall that a semi-monotone operator is monotone in one of its variables, and completely continuous (i.e., continuous from the weak to the strong topology) in the other. It has to be mentioned that the semi-monotone operators have already been studied by J. Leray and J. L. Lions ([14]). Moreover, by our results, Brouwer's fixed point theorem can be easily deduced. Finally, we illustrate our theoretical results by an application in engineering (see [25], [17], [19], [16], [27], [12] for similar approach).

2 The abstract framework

Let V be a real reflexive Banach space endowed with the norm topology, and let V^* be its dual endowed with the weak*-topology. Throughout the paper the duality pairing between a Banach space and its dual is denoted by $\langle .,. \rangle$.We assume that the following statements are valid:

(H1) $C \subseteq V$ is a nonempty subset of V;
(H2) $T : V \to L^p(\Omega, \Re^k)$ is a linear and continuous operator, where $1 \leq p < \infty, k \geq 1$ and $\Omega \subseteq \Re^n$ is a bounded open set in n-dimensional Euclidean space;
(H3) $A : C \times C \rightsquigarrow V^*$ is a set-valued mapping;
The properties of the set-valued mapping A will be given later.
(H4) $j = j(x, y) : \Omega \times \Re^k \to \Re$ is a Caratheodory function, which is locally Lipschitz with respect to the second variable and satisfies the following assumption

$$\begin{aligned} &\exists h_1 \in L^{\frac{p}{p-1}}(\Omega, \Re) \text{ and } h_2 \in L^{\infty}(\Omega, \Re) \text{ such that} \\ &|z| \leq h_1(x) + h_2(x)|y|^{p-1} \text{ a.e. } x \in \Omega, \forall y \in \Re^k, \forall z \in \partial j(x, y) \end{aligned}$$

where

$$j^0(x, y)(h) = \lim \sup_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y' + th) - j(x, y')}{t}$$

is the (partial) Clarke derivative of the locally Lipschitz mapping $j(x, .)$, $x \in \Omega$ fixed, at the point $y \in \Re^k$ with respect to the direction $h \in \Re^k$
and

$$\partial j(x, y) = \left\{ z \in \Re^k : \langle z, h \rangle \leq j^0(x, y)(h), \forall h \in \Re^k \right\}$$

is the Clarke generalized gradient of the mapping $j(x, .)$ at the point $y \in \Re^k$.

We study the following problem:

Find $u \in C$ such that

$$\sup_{f \in A(u,u)} \langle f, v - u \rangle + \int_{\Omega} j^0(x, Tu(x))(Tv(x) - Tu(x))\, dx \geq 0, \forall v \in C. \qquad \text{(P)}$$

Remark. Working on a reflexive Banach space gives us the opportunity to exploit the properties of this particular space, as well as to study the hemivariational inequality problem (P) using minimax results or recession analysis concepts. One important property of a reflexive Banach space is stated below.

Lemma 2.1. *(see [30]): A Banach space is reflexive if and only if it is sequentially weakly compact, i.e., every bounded sequence contains a weakly convergent subsequence.*

To formulate and prove our main theorems, we need to recall some results. The first one, the Inequality of Ky Fan, is of central importance for fixed-point theory, game theory and mathematical economics.

Theorem 2.2. *(see [7]): If the following three assumptions are satisfied:*
(a) the function $f : X \times X \to \Re$ is given where X is a compact, convex, and nonempty set in a topological vector space;

(b) f is quasi-concave in the second argument, i.e., $y \longmapsto f(x,y)$ is quasi-concave on X for every fixed $x \in X$;

(c) f is lower semicontinuous in the first argument, i.e., $x \longmapsto f(x,y)$ is lower semicontinuous on X for every fixed $y \in X$

then

$$\min_{x\in X} \sup_{y\in X} f(x,y) \leq \sup_{x\in X} f(x,x). \tag{2.1}$$

Often the following special case of (2.1) is used. If the above assumptions (a), (b), (c) are satisfied and $f(x,x) \leq 0$ for all $x \in X$, then there exists a $x \in X$ with

$$f(x,y) \leq 0, \text{ for all } y \in X. \tag{2.2}$$

Because the proposed hemivariational inequality problem (P) contains the notion of (partial) Clarke derivative, we need to emphasize some properties of this derivative:

Proposition 2.3. *(see [6],[18]): Let V be a Banach space, $U \subseteq V$ an open subset of V, $f : U \to \Re$ a locally Lipschitz mapping. Then the following statements hold:*

(a) for each $u \in U$, the Clarke derivative $f^0(u)(\cdot) : V \to \Re$ is positive homogeneous and subaditive (then it is convex also) and satisfies

$$\left|f^0(u)(v)\right| \leq K\|v\|, \forall v \in V. \tag{2.3}$$

Moreover, it is Lipschitz continuous on V with Lipschitz constant K, where $K > 0$ is the Lipschitz constant of f around u;

(b) $f^0(\cdot)(\cdot) : U \times V \to \Re$ is upper semicontinuous;

(c) $f^0(u)(-v) = (-f)^0(u)(v), \forall u \in U, \forall v \in V$.

Another important result is given below:

Lemma 2.4. *(see [27]): If the assumption (H4) is satisfied and V_1, V_2 are nonempty subsets of V, then the mapping $V_1 \times V_2 \to \Re$ defined by*

$$(u,v) \in V_1 \times V_2 \longmapsto \int_{\Omega} j^0(x, Tu(x))(Tv(x))\,dx$$

is upper semicontinuous.

We will employ the following lemma related to upper semicontinuity.

Lemma 2.5. *(see [4]): Let X, Y be two topological spaces, let $G : Y \rightsquigarrow X$ be a set-valued mapping, and $g : X \times Y \to \Re$ a real-valued function. Consider the marginal function φ defined by*

$$\varphi : Y \to \Re, \varphi(y) = \sup_{x\in G(y)} g(x,y).$$

If g is upper semicontinuous on $X \times Y$, if there exists $y_0 \in Y$ such that $G(y_0)$ is compact and G is upper semicontinuous at y_0, then φ is also upper semicontinuous at y_0.

Finally, we quote the following Minimax Theorem of Simons ([29]).

Lemma 2.6. *Let X be a topological space, Y a real Banach space, $K \subseteq X$ a convex compact subset of X, $C \subseteq Y$ a convex subset of Y, $\Phi : K \times C \to \Re$ such that*

$\Phi(.,v) : K \to \Re$, where $v \in C$ fixed, is upper semicontinuous on K and is concave on K;
$\Phi(u,.) : C \to \Re$, where $u \in K$ fixed, is convex on C.

Then

$$\inf_{v \in C} \sup_{u \in K} \Phi(u,v) = \sup_{u \in K} \max_{v \in C} \Phi(u,v).$$

Let us recall some definitions.

Definition 2.7. *We say that the set-valued mapping $A : C \rightsquigarrow V^*$ is* **monotone** *if it satisfies the relation*

$$\langle f - g, u - v \rangle \geq 0, \forall u, v \in C, \forall f \in A(u), g \in A(v).$$

Definition 2.8. *If C is a convex set, the set-valued mapping $A : C \rightsquigarrow V^*$ is said to be* **concave** *if*

$$(1-\alpha) A(x_1) + \alpha A(x_2) \supseteq A((1-\alpha)x_1 + \alpha x_2), \forall \alpha \in [0,1], \forall x_1, x_2 \in C.$$

3 The main results

The main results of the paper deal with the existence of the solutions of the problem (P) on real reflexive Banach spaces.

Theorem 3.1. *Let V be a real reflexive Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology.*

Assume that the hypotheses (H1)-(H4) are satisfied and $C \subseteq V$ is a nonempty bounded closed convex subset of V. Moreover, the following assumptions hold:

(i) for each $v \in C$, the set-valued mapping $A(.,v) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous from the line of C into V^*, concave and monotone;

(ii) for each $u \in C$, the set-valued mapping $A(u,.) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii) for each $u, v \in C$, the set $A(u,v)$ is weakly-compact.

Then the problem (P) admits a solution.

If in addition $A(u,u)$ is a convex set, then u is also solution of the following problem:

Find $u \in C, f \in A(u,u)$ such that

$$\langle f, v-u \rangle + \int_{\Omega} j^0(x, Tu(x))(Tv(x) - Tu(x))\, dx \geq 0, \forall v \in V. \qquad (Pc)$$

In the Theorem 3.1, the subset C was bounded. In order to prove a similar result when C is an unbounded set, we refer to the so-called 'recession analysis' (see [2]).

Let us consider a nonempty closed convex subset C of a real reflexive Banach space V.

A vector y is called a **recession direction in C corresponding to the vector x** if

$$\forall t > 0, x + ty \in C.$$

Recession directions are independent of x and they determine a closed convex cone called the **recession cone of C**:

$$C_\infty := \bigcap_{t>0} \left[\frac{C - u_0}{t}\right] \text{ , where } u_0 \in C \text{ is an arbitrarly chosen element.}$$

A useful characterization of the recession cone elements is given below.

$$u \in C_\infty \Longleftrightarrow \exists \left(t_n\right)_{n\in N} \subseteq \Re_+, \exists \left(u_n\right)_{n\in N} \subseteq C \text{ s.t.} \begin{cases} \lim\limits_{n\to\infty} t_n = \infty \\ \lim\limits_{n\to\infty} \dfrac{u_n}{t_n} = u \end{cases}$$

The notion of recession cone has been used to enlarge the results of the classical theory of convexity (see, for instance, [28]).

We define the set $R(A, j, C)$ of **asymptotic directions** by

$$R(A,j,C) = \left\{ \begin{array}{c} w \in C_\infty \text{ such that } \exists \left(u_n\right) \subseteq C, t_n := \|u_n\| \to \infty, w_n := \dfrac{u_n}{\|u_n\|} \rightharpoonup w, \\ \\ \inf\limits_{f\in A(u_n,u_n)} \langle f, u_n\rangle - \int\limits_\Omega j^0\left(x, Tu_n\left(x\right)\right)\left(-Tu_n\left(x\right)\right) dx \leq 0 \end{array} \right\}$$

Hence, we can state the second theorem.

Theorem 3.2. *Let V be a real reflexive Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology.*

Assume that all the hypotheses (H1)-(H4) are satisfied and $C \subset V$ is a nonempty unbounded closed convex subset of V such that $0 \in C$. Moreover,

(i) for each $v \in C$, the set-valued mapping $A(.,v) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous from the line segments of C into V^*, concave and monotone;

(ii) for each $u \in C$, the set-valued mapping $A(u,.) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii) for each $u, v \in C$, the set $A(u,v)$ is weakly-compact;

(iv) $R(A,j,C) = \emptyset$.

Then the problem (P) admits a solution.

If in addition the set $A(u,u)$ is convex, then the problem (Pc) admits solution also.

4 Proofs of the theorems

4.1 Proof of the first theorem

For the proof of Theorem 3.1 let us consider $\Phi : C \times C \to \Re$ be a real-valued mapping defined by

$$\Phi(u,v) = \inf_{g\in A(v,u)} \langle g, u - v\rangle - \int_\Omega j^0\left(x, Tu\left(x\right)\right)\left(Tv\left(x\right) - Tu\left(x\right)\right) dx, \forall u, v \in C.$$

We verify all the assumptions of the Theorem 2.2:
Assumption (a): It is satisfied because of the definitions of Φ.
Assumption (b): For every $u \in C, v \longmapsto \Phi(u,v)$ is a concave mapping.
Indeed, let $u \in C$ be a fixed element.

- Let $\alpha \in [0,1]$, $v_1, v_2 \in C$ chosen arbitrarily. Then using the linearity of T and Proposition 2.3 we have

$$\begin{aligned} & j^0(x, Tu(x))(T(\alpha v_1 + (1-\alpha)v_2)(x) - Tu(x)) \\ &= j^0(x,Tu(x))(\alpha(Tv_1(x) - Tu(x)) + (1-\alpha)(Tv_2(x) - Tu(x))) \\ &\leq \alpha j^0(x,Tu(x))(Tv_1(x) - Tu(x)) + (1-\alpha) j^0(x,Tu(x))(Tv_2(x) - Tu(x)). \end{aligned}$$

  This implies that

$$v \longmapsto -\int_\Omega j^0(x,Tu(x))(Tv(x) - Tu(x))$$

  is a concave mapping.
- Let $\alpha \in [0,1]$, $v_1, v_2 \in C$ chosen arbitrarily. Then, from the fact that the set-valued mapping $v \rightsquigarrow A(v,u)$ is concave and monotone, we have

$$\begin{aligned} & \inf_{f \in A(\alpha v_1 + (1-\alpha)v_2, u)} \langle f, u - (\alpha v_1 + (1-\alpha)v_2)\rangle \\ &\geq \inf_{f \in \alpha A(v_1,u) + (1-\alpha)A(v_2,u)} \langle f, u - (\alpha v_1 + (1-\alpha)v_2)\rangle \\ &= \inf_{\substack{f_1 \in A(v_1,u) \\ f_2 \in A(v_2,u)}} \left\{ \begin{array}{c} \alpha[\langle f_1, u - v_1\rangle + (1-\alpha)\langle f_1, v_1 - v_2\rangle] \\ + (1-\alpha)[\langle f_2, u - v_2\rangle + \alpha\langle f_2, v_2 - v_1\rangle] \end{array} \right\} \\ &= \inf_{\substack{f_1 \in A(v_1,u) \\ f_2 \in A(v_2,u)}} \left\{ \begin{array}{c} \alpha\langle f_1, u - v_1\rangle + (1-\alpha)\langle f_2, u - v_2\rangle \\ +\alpha(1-\alpha)\langle f_1 - f_2, v_1 - v_2\rangle \end{array} \right\} \\ &\geq \inf_{\substack{f_1 \in A(v_1,u) \\ f_2 \in A(v_2,u)}} \{\alpha\langle f_1, u - v_1\rangle + (1-\alpha)\langle f_2, u - v_2\rangle\} \\ &= \alpha \inf_{f_1 \in A(v_1,u)} \langle f_1, u - v_1\rangle + (1-\alpha) \inf_{f_2 \in A(v_2,u)} \langle f_2, u - v_2\rangle. \end{aligned}$$

  This implies that $v \longmapsto \inf_{f \in A(v,u)} \langle f, u - v\rangle$ is a concave mapping.

  As a conclusion, $v \longmapsto \Phi(u,v)$ is a concave mapping.
  Assumption (c): For every $v \in C$, $u \longmapsto \Phi(u,v)$ is lower semicontinuous.
  Indeed, let $v \in C$ a fixed element.

- The mapping $u \longmapsto \int_\Omega j^0(x,Tu(x))(Tv(x) - Tu(x))\,dx$ is upper semicontinuous, according to the Lemma 2.4. Then the mapping

$$u \longmapsto -\int_\Omega j^0(x,Tu(x))(Tv(x) - Tu(x))\,dx$$

  is lower semicontinuous.

- We claim that $u \longmapsto \inf_{g\in A(v,u)} \langle g, u-v\rangle$ is lower semicontinuous. Indeed, we notice that
$$-\inf_{g\in A(v,u)} \langle g, u-v\rangle = \sup_{g\in A(v,u)} \langle g, v-u\rangle .$$
The real-valued mapping $(g,u) \in (V^*, V) \longmapsto \langle g, v-u\rangle \in \Re$ is continuous if V is endowed with the norm topology and V* is endowed with the weak* topology. The set-valued mapping $A(v,.)$ is weakly-upper semicontinuous and the set $A(v,u)$ is weakly-compact. As a consequence Lemma 2.5 implies that the real-valued mapping $u \longmapsto \sup_{g\in A(v,u)} \langle g, v-u\rangle \in \Re$ is upper semicontinuous. Then
$$u \longmapsto \inf_{g\in A(v,u)} \langle g, u-v\rangle$$
is lower semicontinuous.

So we showed that the mapping $u \longmapsto \Phi(u,v)$ is lower semicontinuous.
We have that $\Phi(u,u) = 0$, for all $u \in C$.
Then, by the Theorem 2.2,

$$\text{there exists } u \in C \text{ such that } \Psi(u,v) \le 0, \text{ for all } v \in C$$

i.e.

$$\text{there exists } u \in C :$$
$$\sup_{g\in A(v,u)} \langle g, v-u\rangle + \int_\Omega j^0(x, Tu(x))(Tv(x) - Tu(x))\,dx \ge 0, \forall v \in C. \tag{Pv}$$

Let u be a solution of the problem (Pv) and let $v \in C$ be an arbitrary element.
Denote by $v_t = tv + (1-t)u \in C$, for all $t \in [0,1]$.
According to the hemivariational inequality problem (Pv) one can deduce that:

$$\sup_{g\in A(v_t,u)} \langle g, v_t-u\rangle + \int_\Omega j^0(x, Tu(x))(Tv_t(x) - Tu(x))\,dx \ge 0, \forall t \in [0,1].$$

Since the operator T is linear, we can infer

$$\sup_{g\in A(v_t,u)} \langle g, t(v-u)\rangle + \int_\Omega j^0(x, Tu(x))(t(Tv(x) - Tu(x)))\,dx \ge 0, \forall t \in [0,1].$$

Moreover, because $j^0(x, Tu(x))(.)$ is positive homogeneous, by Proposition 2.3, we obtain

$$\sup_{g\in A(v_t,u)} \langle g, v-u\rangle + \int_\Omega j^0(x, Tu(x))(Tv(x) - Tu(x))\,dx \ge 0, \forall t \in (0,1]. \tag{4.1}$$

We define the set-valued mapping $\begin{cases} G : [0,1] \rightsquigarrow V^* \\ G(t) = A(v_t, u) \end{cases}$ which, according to the assumption (i), is weakly-semicontinuous at zero. Then, knowing that $(g,v) \longmapsto \langle g, v-u\rangle$ is continuous and $G(0) = A(u,u)$ is weakly-compact, by assumption (iii), Lemma 2.5 implies that

$$t \longmapsto \sup_{g \in G(t)} \langle g, v - u \rangle \text{ is upper semicontinuous at zero.}$$

Thus,

$$\limsup_{t \to 0} \sup_{g \in G(t)} \langle g, v - u \rangle \leq \sup_{g \in G(0)} \langle g, v - u \rangle = \sup_{g \in A(u,u)} \langle g, v - u \rangle .$$

Relation (4.1) implies

$$\sup_{g \in A(u,u)} \langle g, v - u \rangle + \int_{\Omega} j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx \geq 0, \forall v \in C,$$

which means that the problem (P) admits a solution.

In the last part of the proof we are concerned with the existence of the solution of the problem (Pc).

For this reason, let us consider u to be a solution of the problem (P). The problem (P) is equivalent with the following:

Find $u \in C$ such that

$$\inf_{v \in C} \sup_{f \in A(u,u)} \left[\langle f, v - u \rangle + \int_{\Omega} j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx \right] \geq 0. \tag{Pm}$$

Let us define the mapping

$$\Psi : A(u,u) \times C \to \Re$$

$$\Psi (f, v) = \langle f, v - u \rangle + \int_{\Omega} j^0 (x, Tu(x)) (Tv(x) - Tu(x)) \, dx$$

where,

$A(u,u)$ is a convex weakly-compact subset of the topological space V^*;

C is a convex subset of the real Banach space V;

$\Psi(.,v) : A(u,u) \to \Re$, with fixed $v \in C$, is continuous on $A(u,u)$ (because of the continuity of the mapping $f \longmapsto \langle f, v - u \rangle$) and it is concave on $A(u,u)$;

$\Psi(f,.) : C \to \Re$, with fixed $f \in A(u,u)$, is convex on C, because the mapping $h \longmapsto j^0 (x, y)(h)$ is convex ([3]).

According to the Lemma 2.6, one can deduce that

$$\inf_{v \in C} \sup_{f \in A(u,u)} \Psi (f, v) = \sup_{f \in A(u,u)} \inf_{v \in C} \Psi (f, v)$$

Due to the fact that $A(u,u)$ is weakly-compact, we can derive that the supremum of the upper-semicontinuous mapping $f \in A(u,u) \longmapsto \inf_{v \in C} \Psi(f,v)$ is attained, i.e.

$$\exists f \in A(u,u) \text{ such that } \inf_{v \in C} \Psi (f, v) = \sup_{f \in A(u,u)} \inf_{v \in C} \Psi (f, v) \geq 0$$

So there exists $f \in A(u,u)$ such that $\Psi(f,v) \geq 0$, for all $v \in C$. That is

$$\exists u \in C, \exists f \in A(u,u) \text{ such that}$$

$$\langle f, v-u \rangle + \int_{\Omega} j^0 (x, Tu(x)) (Tv(x) - Tu(x))\, dx \geq 0, \forall v \in C.$$

This completes the proof. □

4.2 Proof of the second theorem

The proof of the Theorem 3.2 is based upon the Theorem 3.1, and follows the same idea of [1], [2]. We denote by C_n the following set

$$C_n := \{ v \in C : \|v\| \leq n \} \ , \forall n \in N^*.$$

According to the Theorem 3.1 there exists $u_n \in C_n$ such that

$$\sup_{f \in A(u_n,u_n)} \langle f, v-u_n \rangle + \int_{\Omega} j^0 (x, Tu_n(x)) (Tv(x) - Tu_n(x))\, dx \geq 0, \forall v \in C_n.$$

Next, the proof is given in two steps:

- In the first step we claim that there exists $n_0 \in N$ such that $\|u_{n_0}\| < n_0$.
  Indeed, we assume the contrary, i.e., for every $n \in N^*$, $\|u_n\| = n$.
  We define the sequence $(w_n)_{n \in N}$, where $w_n := \frac{u_n}{\|u_n\|}$. According to the Lemma 2.1, it exists a subsequence of the sequence $(w_n)_{n \in N}$, denoted for simplicity also $(w_n)_{n \in N}$ again, such that $w_n \rightharpoonup w$.
  Moreover,

  $$\sup_{f \in A(u_n,u_n)} \langle f, v-u_n \rangle + \int_{\Omega} j^0 (x, Tu_n(x)) (Tv(x) - Tu_n(x))\, dx \geq 0, \forall v \in C_n.$$

  Taking v=0 in the relation above, we get

  $$\sup_{f \in A(u_n,u_n)} \langle f, -u_n \rangle + \int_{\Omega} j^0 (x, Tu_n(x)) (-Tu_n(x))\, dx \geq 0$$

  i.e.,

  $$\inf_{f \in A(u_n,u_n)} \langle f, u_n \rangle - \int_{\Omega} j^0 (x, Tu_n(x)) (-Tu_n(x))\, dx \leq 0.$$

  This implies that $w \in R(A,j,C)$ which is a contradiction with the assumption (iv). Hence, it exists $n_0 \in N$ such that $\|u_{n_0}\| < n_0$.
- In the second step we prove that u_{n_0} is a solution of the problem (P).
  Since $\|u_{n_0}\| < n_0$, then for every $y \in C$, there exists $\varepsilon > 0$ such that $u_{n_0} + \varepsilon (y - u_{n_0}) \in C_{n_0}$.

We can take, for instance,

$$\begin{aligned} &\varepsilon < \frac{n_0 - \|u_{n_0}\|}{\|y - u_{n_0}\|}, \text{ if } y \neq u_{n_0} \\ &\varepsilon = 1, \qquad\qquad\quad \text{if } y = u_{n_0}. \end{aligned}$$

So, we know that for every $v \in C_{n_0}$

$$\sup_{f \in A\left(u_{n_0}, u_{n_0}\right)} \langle f, v - u_{n_0} \rangle + \int_\Omega j^0\left(x, Tu_{n_0}(x)\right)\left(Tv(x) - Tu_{n_0}(x)\right) dx \geq 0.$$

By taking $v = u_{n_0} + \varepsilon\left(y - u_{n_0}\right) \in C_{n_0}$ in the last relation, we obtain that

$$\sup_{f \in A\left(u_{n_0}, u_{n_0}\right)} \langle f, \varepsilon\left(y - u_{n_0}\right) \rangle$$

$$+ \int_\Omega j^0\left(x, Tu_{n_0}(x)\right)\left(\varepsilon\left(Ty(x) - Tu_{n_0}(x)\right)\right) dx \geq 0,$$

because T is a linear operator. Using the fact that the mapping $h \longmapsto j^0(x, y)(h)$ is positive homogeneous, we conclude that

$$\sup_{f \in A\left(u_{n_0}, u_{n_0}\right)} \langle f, y - u_{n_0} \rangle + \int_\Omega j^0\left(x, Tu_{n_0}(x)\right)\left(Ty(x) - Tu_{n_0}(x)\right) dx \geq 0.$$

Because the last inequality holds for every $y \in C$, it follows that the problem (P) admits the solution u_0.

The final part of the proof is based on the same argument as in the proof of Theorem 3.1.
□

5 Applications

In this section, we shall give some applications of our generalized hemivariational inequality.

5.1 Applications to existence theorems of Hartman-Stampacchia type

Let us put ourselves within the framework of [27], where under the hypothesis (H1), (H2), (H4) the following problem is studied with respect to the operator $A : C \subseteq V \to V^*$:

find $u \in C$ such that for every $v \in C$

$$\langle Au, v - u \rangle + \int_\Omega j^0\left(x, Tu(x)\right)\left(Tv(x) - Tu(x)\right) dx \geq 0. \tag{P1}$$

P. D.. Panagiotopoulos, M. Fundo and V. Radulescu proved the following theorem.

Theorem 5.1. *(see [27]): Let V be a real reflexive infinite dimensional Banach space and let $T : V \to L^p\left(\Omega, \Re^k\right)$ be a linear and compact operator. Assume C is a closed, bounded and convex subset of V and $A : C \to V^*$ is monotone and continuous on finite dimensional subspaces of C, i.e., A is demicontinuous. If the function j satisfies the condition (H4), then the problem (P1) has at least one solution.*

We would like to point out that similar results can be obtained from the result of the previous sections with an appropriate choice of operator A. If we apply the Theorem 3.1 to $A : C \to V^*$, which can be viewed as an operator of two variables $A : C \times C \to V^*$, with $A(u, v) = A(v)$, we can deduce the following consequence.

Corollary 5.2. *Let V be a real reflexive Banach space and let $T : V \to L^p\left(\Omega, \Re^k\right)$ be a linear and continuous operator. Assume C is a closed, bounded and convex subset of V and $A : C \to V^*$ is weakly-upper semicontinuous. If the function j satisfies the condition (H4), then the problem (P1) has at least one solution.*

On the other hand, if we look at the operator A as $A : C \times C \to V^*$, with $A(u, v) = A(u)$, we can derive another similar consequence.

Corollary 5.3. *Let V be a real reflexive Banach space and let $T : V \to L^p\left(\Omega, \Re^k\right)$ be a linear and continuous operator. Assume C is a closed, bounded and convex subset of V and $A : C \to V^*$ is affine, monotone and weakly-upper semicontinuous from the lines of C into V^*. If the function j satisfies the condition (H4), then the problem (P1) has at least one solution.*

5.2 Applications to existence theorems for variational inequalities

In a recent paper, Y. Q. Chen extended the classical variational inequality problem for operators with two variables. The following result has been proved.

Theorem 5.4. *(see [5]): Let V be a real reflexive Banach space and $C \subseteq V$ be a bounded closed convex subset. Let $A : C \times C \to V^*$ be a mapping satisfying*

(a) for each $u \in C$, $A(u, .)$ is monotone and demicontinuous;

(b) for each $v \in C$, $A(., v)$ is completely continuous, i.e., it is continuous from V with weak to V^* with strong convergence.

Then the following variational inequality

$$\left(A(w, w), u - w\right) \geq 0, \forall u \in C \tag{P2}$$

has a solution $w \in C$.

Let us remark that this result is a direct consequences of those presented in [11].

We note that complete continuity is so severe a restriction that it never holds for the identity in the infinite-dimensional Hilbert space. If the operator $A(u, .)$ is affine for all $u \in C$, we can lessen the condition (b) of Theorem 5.4, applying again the Theorem 3.1 directly for the operator $A : C \times C \to V^*$, where j is taken to be identically zero. Thus, the result obtained can be stated as follows.

Corollary 5.5. *Let V be a real reflexive Banach space and $C \subseteq V$ be a bounded closed convex subset. Let $A : C \times C \to V^*$ be a mapping satisfying*

(a) for each $u \in C$, $A(u, .)$ is affine, monotone and weakly-upper semicontinuous from the lines of C into V^*;

(b) for each $v \in C$, $A(., v)$ is weakly-upper semicontinuous.

Then the following variational inequality (P2) has a solution $w \in C$.

5.3 Application to Brouwer's fixed point theorem

Let us recall the Brouwer's fixed theorem.

Theorem 5.6. *Let $C \subseteq \Re^n$ be a nonempty compact convex set and $f : C \to C$ continuous. Then f has a fixed point.*

As we have mentioned in Introduction, from our results Brouwer's fixed point theorem can be easily deduced. Indeed, in order to show that f admits a fixed point, let $V = \Re^n$ and

$$A : C \times C \to V$$
$$A(u, v) = u - f(v).$$

It is easy to verify that this map satisfies all the assumptions of the Theorem 3.1, where j is taken to be identically zero. We can infer that there exists an element $u \in C$ such that

$$\langle u - f(u), v - u \rangle \geq 0, \forall v \in C.$$

If we put $v = f(u) \in C$ in the above inequality, then we immediately obtain

$$f(u) = u$$

i.e., u is a fixed point of f.

5.4 Applications to Engineering

Our results can be applied directly to the study of B. V. P.s in mechanics and engineering. Let us analyze a very general situation which leads us to the hemivariational inequality problem (P). For instance, let us consider an open, bounded, connected subset $\Omega \subseteq \Re^3$ referred to a fixed Cartesian coordinate system $Ox_1x_2x_3$ and we formulate the problem

$$-\Delta u + h(u) = w \text{ in } \Omega \tag{5.1}$$

$$u = 0 \text{ on } \Gamma. \tag{5.2}$$

Here Γ is the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient) and h is a continuous function. Moreover, u may represent the temperature in the case of heat conduction problems, whereas in problems of hydraulics and electrostatics the pressure and the electric potential are represented, respectively. See for instance [9] for a comprehensive material about mathematical modeling.

We seek a function u such that to verify (5.1), (5.2) with

$$-w \in \partial j(x,u) \tag{5.3}$$

where the function w is known, $j : \Omega \times \Re \to \Re$ satisfies the assumption (H4), and $\partial j(x,y)$ denotes the Clarke generalized gradient of the mapping $j(x,.)$ at the point $y \in \Re$.

Let us consider the Sobolev space $V = H_0^1(\Omega)$. We multiply (5.1) by $(v-u)$ and integrate over Ω. This gives us the following relation

$$\int_\Omega -\Delta u (v-u)\, dx + \int_\Omega h(u)(v-u)\, dx = \int_\Omega w(v-u)\, dx. \tag{5.4}$$

Then from the Gauss-Green Theorem applied to (5.4) we are led to the equality

$$\int_\Omega \nabla u \nabla (v-u)\, dx + \int_\Omega h(u)(v-u)\, dx = \int_\Omega w(v-u)\, dx. \tag{5.5}$$

Because $u, v \in H_0^1(\Omega)$ the surface integral vanished.

We may ask in addition that u is constrained to belong to a convex bounded closed set $C \subseteq V$ due to some technical reasons, e.g., constraints for the temperature or the pressure of the fluid, etc.

Relation (5.3) and the definition of the Clarke generalized gradient imply that

$$-w(v-u) \leq j^0(x,u)(v-u). \tag{5.6}$$

Then the relations (5.5) and (5.6) give the inequality

$$\int_\Omega \nabla u \nabla (v-u)\, dx + \int_\Omega h(u)(v-u)\, dx + \int_\Omega j^0(x,u)(v-u)\, dx \geq 0, \forall v \in C. \tag{5.7}$$

Let us note that there exist a linear monotone continuous operator $B : C \to V^*$ and a continuous operator $D : C \to V^*$ such that

$$\langle B(u), v\rangle = \int_\Omega \nabla u \nabla v dx, \forall u, v \in V$$

and

$$\langle D(u), v\rangle = \int_\Omega uv dx, \forall u, v \in V.$$

Thus, if we consider the following multivalued mapping

$$A : C \times C \rightsquigarrow V^*$$
$$A(u,v) = B(u) + D(h(v))$$

then the hemivariational inequality (5.7) leads us to the following problem:

find $u \in C$ such that for any $v \in C$

$$\sup_{f \in A(u,u)} \langle f, v-u\rangle + \int_\Omega j^0(x,u)(v-u)\, dx \geq 0. \tag{P3}$$

Since the multivalued operator A satisfies the assumptions (i), (ii), (iii) of the Theorem 3.1 and the embedding of V in $L^2(\Omega)$ is linear and continuous, we can prove the existence of solutions of (P3) by simply applying our above mentioned result.

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