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CONTROLLABILITY FOR SEMILINEAR FUZZY INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, the controllability for the semilinear fuzzy integrodifferential equation with nonlocal condition is established via Banach fixed point analysis approach using the fuzzy number whose values are normal, convex upper semicontinuous and compactly supported interval in E_N.

Keywords: fuzzy number, controllability, integrodifferential equation, fuzzy solution, Banach fixed point theorem.

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1 Introduction

In real life models, many systems are related to 'uncertainty' and/or 'inexactness'. The problem of 'inexactness' is considered in general exact science, whereas that of 'uncertainty' is vagueness or fuzzy and accidental. Ding and Kandel [10] developed a way to combine differential equations with fuzzy sets to form a fuzzy logic system called fuzzy dynamical systems, which can be utilized to form a fuzzy semilinear integrodifferential equation.

For fuzzy concepts recently Diamond and Kloeden [9] and Subrahmanyam and Sudarsanam [23] established a metric space theory for fuzzy sets and fuzzy Volterra integral equations, respectively. In particular, Kaleva [14] studied fuzzy differential equations, the Cauchy problem for continuous fuzzy differential equations was studied by Nieto [18], and Song *et al* [22] obtained results for global solutions of such systems.

Seikkala [21] proved the existence and uniqueness of a fuzzy solution for the following systems

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0,$$

where f is a continuous mapping from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$ and x_0 is a fuzzy number. Recently, the above concept has been extended to integrodifferential equations by Balasubramaniam and Muralisankar [4]. Nonlocal boundary conditions yield better results than the classical condition $x(0) = x_0$, and applications of this are well documented (for details see [3], [5], [6], [19] and the references therein).

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite and infinite dimensional spaces has been extensively studied (e.g., see [1], [7], [8]). Quinn and Carmichael [20] have shown that the controllability problem in Banach spaces can be converted into a fixed point problem for a single-valued mapping. Using the fixed point technique, the nonlocal controllability of nonlinear integrodifferential systems has been studied by Balasubramaniam and Loganathan [2]. Recently, the performance of the fuzzy logic system (for example optimal control, observability) from the aspect of fuzzy differential equations has attracted several researchers. In particular, Kwun *et. al.* [15] studied the existence of fuzzy optimal controls for nonlinear fuzzy differential systems with a nonlocal condition. Further, Ding *et. al.* [11] analyzed a way to combine differential equations with fuzzy sets to form a fuzzy logic system called a fuzzy dynamical system and studied the observability of such linear systems.

In the Section 3, instead of using a fuzzy logic approach with a fixed point, the controllability of the following semilinear fuzzy integrodifferential equation with nonlocal boundary condition is studied

$$\dot{x}(t) = \mathsf{A}[x(t) + \int_0^t G(t-s)x(s)ds] + f(t, x(t)), \qquad t \in J, \tag{1}$$

$$x(0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0 \in E_N,$$

where $J = [0, T]$, $\mathsf{A} : J \to E_N$ is a fuzzy coefficient, and E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals. Here it is assumed that $f : J \times E_N \to E_N$ is a nonlinear continuous function and $G(t)$ is an $n \times n$ continuous matrix function such that it's derivative $G'(t)x$ is continuous for $x \in E_N$ and $t \in \mathrm{J}$ with norm $\|G(t)\| \leq k$, $k > 0$. The function $g : J^p \times E_N \to E_N$, is a nonlinear and continuous. The elements of the set $\{t_1, t_2, \dots, t_p\}$, $0 < t_1 < t_2 < \cdots < t_p \leq T$, $p \in N$ are represented by the placeholder '$\cdot$' in the function $x(\cdot)$.

Now, consider the linearly perturbed system

$$\dot{x}(t) = \mathsf{A}[x(t) + \int_0^t G(t-s)x(s)ds] + f(t, x(t)) + \mathsf{B}(t)u(t), \qquad t \in J, \tag{2}$$

$$x(0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0 \in E_N,$$

where $x(t)$ is the state, $u : J \to E_N$ is a control function, and $\mathsf{B} : J \to E_N$ is a fuzzy coefficient.

This class of equations arises, for example, in problems such as heat conduction in materials with memory or population dynamics for spatially distributed populations (see [12], [13], [16]). For such reasons, there has been an increasing interest in the study of systems that can be described by integrodifferential equations.

2 Preliminaries

A fuzzy subset of $\mathbb{R}^n$ is defined in terms of a membership function which assigns to each point $x \in \mathbb{R}^n$ a grade of membership in the fuzzy set. Such a membership function is denoted by

$$v : \mathbb{R}^n \to [0, 1].$$

Throughout this paper it is assumed that v maps $\mathbb{R}^n$ onto $[0, 1]$, $[v]^0$ is a bounded subset of $\mathbb{R}^n$, v is upper semicontinuous and v is fuzzy convex. Denote by E^n the space of all fuzzy subsets v of $\mathbb{R}^n$ that are normal, fuzzy convex, and upper semicontinuous fuzzy sets with bounded supports. In particular, E^1 denote the space of all fuzzy subsets v of $\mathbb{R}$.

A fuzzy number a in real line $\mathbb{R}$ is a fuzzy set characterized by a membership function μ_a as

$$\mu_a : \mathbb{R} \to [0,1].$$

Such a fuzzy number a is expressed as

$$a = \int_{x \in \mathbb{R}} \mu_a(x)/x$$

with the understanding that $\mu_a(x) \in [0,1]$ represents the grade of membership of x in a and $\int$ denotes the union of $\mu_a(x)/x$ over $x \in \mathbb{R}$.

Definition 2.1. *A fuzzy number a in* $\mathbb{R}$ *is said to be convex if for any real numbers* x, y, z *in* $\mathbb{R}$ *with* $x \leq y \leq z$,

$$\mu_a(y) \geq \min\{\mu_a(x), \mu_a(z)\}.$$

Definition 2.2. *The height of a fuzzy set is the largest membership value attained by any point.*

Definition 2.3. *If the height of a fuzzy set equals one, then the fuzzy set is called a normal fuzzy set. Thus, a fuzzy number a in* $\mathbb{R}$ *is called normal if the following holds*

$$\max_x \mu_a(x) = 1.$$

Result 2.4. *Let* E_N *be the set of all upper semicontinuous convex normal fuzzy numbers with bounded* α*-level intervals (see [17]). This means that if a* $\in E_N$*, then the* α*-level set*

$$[a]^\alpha = \{x \in \mathbb{R} : a(x) \geq \alpha,\ 0 < \alpha \leq 1\}$$

is a closed bounded interval, that is denoted by

$$[a]^\alpha = [a_q^\alpha,\ a_r^\alpha],$$

and there exists a $t_0 \in \mathbb{R}$ *such that* $a(t_0) = 1$.

Result 2.5. *Two fuzzy numbers a and b are called equal* $a = b$, *if* $\mu_a(x) = \mu_b(x)$ *for all* $x \in \mathbb{R}$. *It follows that*

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \text{ for all } \alpha \in (0,1].$$

Result 2.6. *A fuzzy number a may be decomposed into its level sets through the resolution identity*

$$a = \int_0^1 \alpha[a]^\alpha,$$

where $\alpha[a]^\alpha$ *is the product of a scalar* α *with the set* $[a]^\alpha$ *and* $\int$ *is the union of* $[a]^\alpha s$ *with the number a ranging from 0 to 1.*

Definition 2.7. *The support of a fuzzy set A in the universal set U is a crisp set that contains all the elements of U that have nonzero membership values in A, that is,*

$$supp(A) = \{x \in U : \mu_a(x) > 0\},$$

where $supp(A)$ denotes the support of fuzzy set A. Hence, the support Γ_a of a fuzzy number a is defined, as a special case of level set, by the following

$$\Gamma_a = \{x : \mu_a(x) > 0\}.$$

Definition 2.8. *A fuzzy number a in $\mathbb{R}$ is said to be positive if $0 < a_1 < a_2$ holds for the support $\Gamma_a = [a_1, a_2]$ of a, that is, Γ_a is in the positive real line. Similarly, a is called negative if $a_1 \leq a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$.*

Lemma 2.9. *([21]) If $a, b \in E_N$, then for $\alpha \in (0,1]$,*

$$\begin{aligned}[a+b]^\alpha &= [a_q^\alpha + b_q^\alpha, a_r^\alpha + b_r^\alpha],\\ [a \cdot b]^\alpha &= [\min\{a_i^\alpha b_j^\alpha\},\ \max\{a_i^\alpha b_j^\alpha\}],\quad (i, j = q, r),\\ [a-b]^\alpha &= [a_q^\alpha - b_r^\alpha, a_r^\alpha - b_q^\alpha].\end{aligned}$$

Lemma 2.10. *([21]) Let $[a_q^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If*

$$[a_q^\beta, a_r^\beta] \subset [a_q^\alpha, a_r^\alpha] \ \text{ for } 0 < \alpha \leq \beta, \tag{3}$$

and

$$[\lim_{k\to\infty} a_q^{\alpha_k}, \lim_{k\to\infty} a_r^{\alpha_k}] = [a_q^\alpha, a_r^\alpha], \tag{4}$$

whenever (α_k) is nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $[a_q^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, consists of the α-level sets of a fuzzy number $a \in E_N$. Conversely, if $[a_q^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number $a \in E_N$, then conditions (3) and (4) are valid.

Let x be a point in $\mathbb{R}^n$ and A be a nonempty subset of $\mathbb{R}^n$. We define the distance $d(x, A)$ from x to the set A by

$$d(x, A) = \inf\{\|x - a\| :\ a \in A\}. \tag{5}$$

Now let A and B be nonempty subsets of $\mathbb{R}^n$. Define the Hausdorff separation of B from A by

$$d_H^*(B, A) = \sup\{d(b, A) :\ b \in B\}. \tag{6}$$

In general,

$$d_H^*(A, B) \neq d_H^*(B, A).$$

Define the Hausdorff distance between nonempty subsets of A and B in $\mathbb{R}^n$ by

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}, \tag{7}$$

which is symmetric in A and B. Consequently,

(1) $d_H(A,B) \geq 0$ with $d_H(A,B) = 0$ if $\bar{A} = \bar{B}$,
(2) $d_H(A,B) = d_H(B,A)$,
(3) $d_H(A,B) \leq d_H(A,C) + d_H(C,B)$,

for any nonempty subsets of A, B and C in $\mathbb{R}^n$. The Hausdorff distance (7) is a metric, called the Hausdorff metric.

The supremum metric d_∞ on E^n is defined by

$$d_\infty(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \ \alpha \in (0,1]\} \tag{8}$$

for all $u, v \in E^n$, and is obviously a metric on E^n.

The supremum metric H_1 on $C(J, E_n)$ is defined by

$$H_1(x,y) = \sup\{d_\infty(x(t), y(t)) : \ t \in J\} \tag{9}$$

for all $x, y \in C(J, E^n)$.

3 Controllability Results

Let I be a real interval. A mapping $x : I \in E_N$ is called a fuzzy process. Denote

$$[x(t)]^\alpha = [x_q^\alpha(t), \ x_r^\alpha(t)], \qquad t \in I, \ 0 < \alpha \leq 1.$$

Then the derivative $\dot{x}(t) \in E_N$ of a fuzzy process x is such that

$$[\dot{x}(t)]^\alpha = [\dot{x}_q^\alpha(t), \dot{x}_r^\alpha(t)], \qquad 0 < \alpha \leq 1. \tag{10}$$

The fuzzy integral

$$\int_a^b x(t)dt, \qquad a, b \in I$$

is defined by

$$\Big[\int_a^b [x(t)dt\Big]^\alpha = \Big[\int_a^b x_q^\alpha(t)dt, \int_a^b x_r^\alpha(t)dt\Big]$$

provided that the Lebesgue integrals on the right exist.

The relation between fuzzy derivative and fuzzy integral is obvious; first

$$\frac{d}{dt}\int_a^t x(s)ds = x(t), \ \ a.e. \ \ t \in I,$$

and if the end point functions $(\dot{x}_q^\alpha)$ and $(\dot{x}_r^\alpha)$ in (10) are integrable, then

$$x(t) = x(a) + \int_a^t \dot{x}(s)ds, \qquad t \in I.$$

Definition 3.1. *The fuzzy process* $x : J \to E_N$ *is a solution of equation* (1) *without the inhomogeneous term if and only if*

$$(\dot{x}_q^\alpha)(t) = \min\Big\{a_i^\alpha(t)\Big[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\Big], i,j = q,r\Big\},$$

$$(\dot{x}_r^\alpha)(t) = \max\Big\{a_i^\alpha(t)\Big[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\Big], i,j = q,r\Big\},$$

and

$$(x_q^\alpha)(0) = x_{0q}^\alpha - g_q^\alpha(t_1, t_2, ..., t_p, x(\cdot)),$$
$$(x_r^\alpha)(0) = x_{0r}^\alpha - g_r^\alpha(t_1, t_2, ..., t_p, x(\cdot)).$$

For the existence and uniqueness of the fuzzy solution

$$x(t) = S(t)\Big(x_0 - g(t_1, t_2, \ldots, t_p, x(\cdot))\Big) + \int_0^t S(t-s)f(s, x(s))ds \tag{11}$$

for the nonlinear fuzzy integrodifferential equations (1), assume the following hypotheses.

- $(H1)$ The nonlinear function $g : J^p \times E_N \to E_N$ is continuous and satisfies the inequality

$$d_H\Big([g(t_1, t_2, \ldots, t_p, \xi(\cdot)))]^\alpha, [g(t_1, t_2, \ldots, t_p, \zeta(\cdot))]^\alpha\Big) \le c_1 d_H\Big([\xi(\cdot)]^\alpha, [\zeta(\cdot)]^\alpha\Big)$$

  for all $\xi(\cdot), \zeta(\cdot) \in E_N$, where c_1 is a finite positive constant.
- $(H2)$ The inhomogeneous term $f : J \times E_N \in E_N$ is a continuous function that satisfies the global Lipschitz condition,

$$d_H\Big([f(s, \xi(s)]^\alpha, [f(s, \zeta(s)]^\alpha\Big) \le c_2 d_H\Big([\xi(s)]^\alpha, [\zeta(s)]^\alpha\Big)$$

  for all $\xi(s)$, $\zeta(s) \in E_N$ and some finite constant $c_2 > 0$.
- $(H3)$ $S(t)$ is a fuzzy number satisfying for $y \in E_N$, $S'(t)y \in C^1(J, E_N) \cap C(J, E_N)$ the equation

$$\begin{aligned}\tfrac{d}{dt}S(t)y &= A\Big[S(t)y + \textstyle\int_0^t G(t-s)S(s)yds\Big]\\ &= S(t)Ay + \int_0^t S(t-s)AG(s)yds,\ t \in J,\end{aligned}$$

  such that

$$[S(t)]^\alpha = [S_q^\alpha(t), S_r^\alpha(t)]$$

  and $S_i^\alpha(t)$ are bounded on J, (i.e., exists a constant $c > 0$ such that $|S_i^\alpha(t)| \le c$ for all t $\in$ J).

For the controllability result of equation (2) with the assumptions $(H1)$–$(H3)$, assume the additional hypothesis.

$(H4)$ The fuzzy map $W : P(\mathbb{R}) \to E_N$ defined by

$$W^\alpha(u) = \begin{cases} \displaystyle\int_0^t S^\alpha(T-s)\mathsf{B}^\alpha(s)u(s)ds, & \text{for } u(s) \subset \bar{\Gamma}_u, \\ 0, & \text{otherwise,} \end{cases}$$

is such that there exists W_i^α $(i = q, r)$ such that

$$W_q^\alpha(u_q) = \int_0^T S_q^\alpha(T-s)\mathsf{B}_q^\alpha(s)u_q(s)ds, \quad u_q(s) \in [u_q^\alpha(s), u_l(s)],$$

$$W_r^\alpha(u_r) = \int_0^T S_r^\alpha(T-s)\mathsf{B}_r^\alpha(s)u_r(s)ds, \quad u_r(s) \in [u_l(s), u_r^\alpha(s)].$$

for all $u \in P(\mathbb{R})$, the set of all closed compact control functions in $\mathbb{R}$ and W_q^α, W_r^α are bijective mappings.

Under hypothesis $(H1)$–$(H3)$, equation (2) is related to the following fuzzy integral system

$$x(t) = S(t)\Big(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\Big) + \int_0^t S(t-s)\Big[f(s, x(s)) + \mathsf{B}(s)u(s)\Big]ds \tag{12}$$

where $S(t)$ is a fuzzy number.

Definition 3.2. *The equation (2) is said to be* **α-level exact controllable** *if, for every x_0, x_1, $g \in E_N$, there exists a control $u : J \to E_N$ such that the fuzzy solution $x(t)$ of (2) satisfies*

$$\Big[x(0) - g(t_1, t_2, \dots, t_p, x(\cdot))\Big]^\alpha = [x_0]^\alpha$$

and $[x(T)]^\alpha = [x_1]^\alpha$ in time T.

Theorem 3.3. *Suppose that hypotheses $(H1)$–$(H4)$ are satisfied and that $2cc_1 < 1$. Then system (2) is α-level exact controllable on J.*

PROOF: Using the bijective mappings in hypotheses $(H4)$ introduce the nonlinear control u(s) as

$$\begin{aligned}[u(s)]^\alpha &= [u_q^\alpha(s), u_r^\alpha(s)] \\ &= \Big[(W_q^\alpha)^{-1}\Big\{(x_1)_q^\alpha - S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1, t_2, ..., t_p, x(\cdot))\Big) - \int_0^T S_q^\alpha(T-s)f_q^\alpha(s, x(s))ds\Big\}, \\ &\quad (W_r^\alpha)^{-1}\Big\{(x_1)_r^\alpha - S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1, t_2, ..., t_p, x(\cdot))\Big) - \int_0^T S_r^\alpha(T-s)f_r^\alpha(s, x(s))ds\Big\}\Big].\end{aligned}$$

Then substituting this expression into equation (12) yields $x(T)$.

$$\begin{aligned}
[x(T)]^\alpha = \Big[& S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) + \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds \\
& + \int_0^T S_q^\alpha(T-\eta)\mathsf{B}_q^\alpha(\eta)(W_q^\alpha)^{-1}\Big\{(x_1)_q^\alpha - S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p),x(\cdot))\Big) \\
& \qquad - \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds\Big\}(\eta)d\eta, \\
& S_r^\alpha(t)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) + \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds \\
& + \int_0^T S_r^\alpha(T-\eta)\mathsf{B}_r^\alpha(\eta)(W_r^\alpha)^{-1}\Big\{(x_1)_r^\alpha - S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) \\
& \qquad - \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds\Big\}(\eta)d\eta\Big] \\
= \Big[& S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) + \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds \\
& +(W_q^\alpha)(W_q^\alpha)^{-1}\Big\{(x_1)_q^\alpha - S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) \\
& \qquad - \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds\Big\}, \\
& S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,...,t_p,x(\cdot))\Big) + \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds \\
& +(W_r^\alpha)(W_r^\alpha)^{-1}\Big\{(x_1)_r^\alpha - S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) \\
& \qquad - \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds\Big\}\Big] \\
= {} & [(x_1)_q^\alpha,(x_1)_r^\alpha] \\
= {} & [x_1]^\alpha.
\end{aligned}$$

Define

$$\begin{aligned}
(\Phi\xi)(t) = {} & S(t)\Big(x_0 - g(t_1,t_2,...,t_p,\xi(\cdot))\Big) + \int_0^t S(t-s) f(s,\xi(s))ds \\
& + \int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\Big\{x_1 - S(T)(x_0 - g(t_1,t_2,...,t_p,\xi(\cdot))\Big) \\
& \qquad - \int_0^T S(T-s) f(s,\xi(s))ds\Big\}(\eta)d\eta.
\end{aligned}$$

It is easy to verify that Φ is a continuous function from $C(J;E_N)$ to itself. For $\xi,\eta \in C(J;E_N)$

$$\begin{aligned}
[x(T)]^\alpha = \Big[& S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p,x(\cdot))\Big) + \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds \\
& + \int_0^T S_q^\alpha(T-\eta)\mathsf{B}_q^\alpha(\eta)(W_q^\alpha)^{-1}\Big\{(x_1)_q^\alpha - S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\ldots,t_p),x(\cdot))\Big) \\
& - \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds\Big\}(\eta)d\eta,
\end{aligned}$$

$$\begin{aligned}
&S_r^\alpha(t)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) + \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds \\
&\qquad + \int_0^T S_r^\alpha(T-\eta)\mathsf{B}_r^\alpha(\eta)(W_r^\alpha)^{-1}\Big\{(x_1)_r^\alpha - S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) \\
&\qquad\quad - \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds\Big\}(\eta)d\eta\Big] \\
&= \Big[S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) + \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds \\
&\qquad +(W_q^\alpha)(W_q^\alpha)^{-1}\Big\{(x_1)_q^\alpha - S_q^\alpha(T)\Big(x_{0q}^\alpha - g_q^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) \\
&\qquad\quad - \int_0^T S_q^\alpha(T-s) f_q^\alpha(s,x(s))ds\Big\}, \\
&\quad S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) + \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds \\
&\qquad +(W_r^\alpha)(W_r^\alpha)^{-1}\Big\{(x_1)_r^\alpha - S_r^\alpha(T)\Big(x_{0r}^\alpha - g_r^\alpha(t_1,t_2,\dots,t_p,x(\cdot))\Big) \\
&\qquad\quad - \int_0^T S_r^\alpha(T-s) f_r^\alpha(s,x(s))ds\Big\}\Big] \\
&= [(x_1)_q^\alpha, (x_1)_r^\alpha] \\
&= [x_1]^\alpha.
\end{aligned}$$

Define

$$\begin{aligned}
(\Phi\xi)(t) &= S(t)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big) + \int_0^t S(t-s)f(s,\xi(s))ds \\
&\quad + \int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\Big\{x_1 - S(T)(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big) \\
&\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}(\eta)d\eta.
\end{aligned}$$

It is easy to verify that Φ is a continuous function from C (J;E_N) to itself. For $\xi, \eta \in C(J;E_N)$

$$\begin{aligned}
&d_H\Big([(\Phi\xi)(t)]^\alpha, [(\Phi\zeta)(t)]^\alpha\Big) \\
&\quad = d_H\Big(\Big[S(t)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big) + \int_0^t S(t-s)f(s,\xi(s))ds \\
&\qquad + \int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}(\eta)d\eta\Big]^{\alpha},\\
&\quad \Big[S(t)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big) + \int_0^t S(t-s)f(s,\zeta(s))ds\\
&\qquad + \int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big)\\
&\qquad\quad - \int_0^T S(T-s)f(s,\zeta(s))ds\Big\}(\eta)d\eta\Big]^{\alpha}\Big)\\
&= d_H\Big(\Big[S(t)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big)\Big]^{\alpha} + \Big[\int_0^t S(t-s)f(s,\xi(s))ds\Big]^{\alpha}\\
&\qquad + \Big[\int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big)\\
&\qquad\quad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}(\eta)d\eta\Big]^{\alpha},\\
&\qquad \Big[S(t)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big)\Big]^{\alpha} + \Big[\int_0^t S(t-s)f(s,\zeta(s))ds\Big]^{\alpha}\\
&\qquad + \Big[\int_0^t S(t-\eta)\mathsf{B}(\eta)W^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big)\\
&\qquad\qquad - \int_0^T S(T-s)f(s,\zeta(s))ds\Big\}(\eta)d\eta\Big]^{\alpha}\Big)\\
&\le d_H\Big([S(t)x_0]^{\alpha} + [S(t)g(t_1,t_2,\dots,t_p,\xi(\cdot))]^{\alpha} + \Big[\int_0^t S(t-s)f(s,\xi(s))ds\Big]^{\alpha},\\
&\qquad [S(t)x_0]^{\alpha} + [S(t)g(t_1,t_2,\dots,t_p,\zeta(\cdot))]^{\alpha} + \Big[\int_0^t S(t-s)f(s,\zeta(s))ds\Big]^{\alpha}\Big)\\
&\qquad + d_H\Big(\Big[\int_0^t S_q^{\alpha}(t-\eta)\mathsf{B}_q^{\alpha}(\eta)(W_q^{\alpha})^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big)\\
&\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}_q^{\alpha}(\eta)d\eta,\\
&\quad \int_0^t S_r^{\alpha}(t-\eta)\mathsf{B}_r^{\alpha}(\eta)(W_r^{\alpha})^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\xi(\cdot))\Big)\\
&\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}_r^{\alpha}(\eta)d\eta\Big],\\
&\quad \Big[\int_0^t S_q^{\alpha}(t-\eta)\mathsf{B}_q^{\alpha}(\eta)(W_q^{\alpha})^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big)\\
&\qquad - \int_0^T S(T-s)f(s,\zeta(s))ds\Big\}_q^{\alpha}(\eta)d\eta,\\
&\quad \int_0^t S_r^{\alpha}(t-\eta)\mathsf{B}_r^{\alpha}(\eta)(W_r^{\alpha})^{-1}\Big\{x_1 - S(T)\Big(x_0 - g(t_1,t_2,\dots,t_p,\zeta(\cdot))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^T S(T-s)f(s,\zeta(s))ds\Big\}_r^\alpha(\eta)d\eta\Big]\Big) \\
&\le d_H\Big(\Big[S_q^\alpha(t)g_q^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot)),S_r^\alpha(t)g_r^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big], \\
&\qquad\qquad \Big[S_q^\alpha(t)g_q^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot)),S_r^\alpha(t)g_r^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))\Big]\Big) \\
&\quad + \int_0^t d_H\Big([S_q^\alpha(t-s)f_q^\alpha(s,\xi(s)),S_r^\alpha(t-s)f_r^\alpha(s,\xi(s))], \\
&\qquad\qquad \Big[S_q^\alpha(t-s)f_q^\alpha(s,\zeta(s)),S_r^\alpha(t-s)f_r^\alpha(s,\zeta(s))\Big]\Big)ds \\
&\quad + d_H\Big(\Big[(W_q^\alpha)(W_q^\alpha)^{-1}\Big\{x_1-S(T)\Big(x_0-g(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big) \\
&\qquad\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}_q^\alpha(\eta)d\eta, \\
&\qquad\quad (W_r^\alpha)(W_r^\alpha)^{-1}\{x_1-S(T)(x_0-g(t_1,t_2,\ldots,t_p,\xi(\cdot))) \\
&\qquad\qquad - \int_0^T S(T-s)f(s,\xi(s))ds\Big\}_r^\alpha(\eta)d\eta\Big], \\
&\qquad\quad \Big[(W_q^\alpha)(W_q^\alpha)^{-1}\{x_1-S(T)(x_0-g(t_1,t_2,\ldots,t_p,\zeta(\cdot))) \\
&\qquad\qquad - \int_0^T S(T-s)f(s,\zeta(s))ds\}_q^\alpha(\eta)d\eta, \\
&\qquad\quad (W_r^\alpha)(W_r^\alpha)^{-1}\{x_1-S(T)(x_0-g(t_1,t_2,\ldots,t_p,\zeta(\cdot))) \\
&\qquad\qquad - \int_0^T S(T-s)f(s,\zeta(s))ds\}_r^\alpha(\eta)d\eta\Big]\Big) \\
&\le \max\Big(\Big|S_q^\alpha(t)\Big[g_q^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))-g_q^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|, \\
&\qquad\qquad \Big|S_r^\alpha(t)\Big[g_r^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))-g_r^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|\Big) \\
&\quad + \int_0^t \max\Big(\Big|S_q^\alpha(t-s)\Big[f_q^\alpha(s,\zeta(s))-f_q^\alpha(s,\xi(s))\Big]\Big|, \\
&\qquad\qquad \Big|S_r^\alpha(t-s)\Big[f_r^\alpha(s,\zeta(s))-f_r^\alpha(s,\xi(s))\Big]\Big|\Big)ds \\
&\quad + \max\Big(\Big|S_q^\alpha(T)\Big[g_q^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))-g_q^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|, \\
&\qquad\qquad \Big|S_r^\alpha(T)\Big[g_r^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))-g_r^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|\Big) \\
&\quad + \int_0^T \max\Big(\Big|S_q^\alpha(T-s)\Big[f_q^\alpha(s,\zeta(s))-f_q^\alpha(s,\xi(s))\Big]\Big|, \\
&\qquad\qquad \Big|S_r^\alpha(T-s)\Big[f_r^\alpha(s,\zeta(s))-f_r^\alpha(s,\xi(s))\Big]\Big|\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 2c\ \max\ \Big(\Big|\Big[g_q^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot)) - g_q^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|, \\
& \qquad\qquad \Big|\Big[g_r^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot)) - g_r^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big]\Big|\Big) \\
& \quad + c\ \int_0^t \max\Big(\Big|f_q^\alpha(s,\zeta(s)) - f_q^\alpha(s,\xi(s))\Big|, \Big|f_r^\alpha(s,\zeta(s)) - f_r^\alpha(s,\xi(s))\Big|\Big)ds \\
& \quad + c\ \int_0^T \max\Big(\Big|f_q^\alpha(s,\zeta(s)) - f_q^\alpha(s,\xi(s))\Big|, \Big|f_r^\alpha(s,\zeta(s)) - f_r^\alpha(s,\xi(s))\Big|\Big)ds \\
=\ & 2c\, d_H\Big(\Big[g_q^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot)), g_r^\alpha(t_1,t_2,\ldots,t_p,\xi(\cdot))\Big], \\
& \qquad\qquad \Big[g_q^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot)) - g_r^\alpha(t_1,t_2,\ldots,t_p,\zeta(\cdot))\Big]\Big) \\
& \quad + c\int_0^t d_H\Big([f_q^\alpha(s,\xi(s)), f_r^\alpha(s,\xi(s))], [f_q^\alpha(s,\zeta(s)), f_r^\alpha(s,\zeta(s))]\Big)ds \\
& \quad + c\int_0^T d_H\Big([f_q^\alpha(s,\xi(s)), f_r^\alpha(s,\xi(s))], [f_q^\alpha(s,\zeta(s)), f_r^\alpha(s,\zeta(s))]\Big)ds \\
=\ & 2c\ d_H\Big([g(t_1,t_2,\ldots,t_p,\xi(\cdot))]^\alpha, [g(t_1,t_2,\ldots,t_p,\zeta(\cdot))]^\alpha\Big) \\
& \quad + c\int_0^t d_H\Big([f(s,\xi(s))]^\alpha, [f(s,\zeta(s))]^\alpha\Big)ds \\
& \quad + c\int_0^T d_H\Big([f(s,\xi(s))]^\alpha, [f(s,\zeta(s))]^\alpha\Big)ds \\
\leq\ & 2cc_1\ d_H\Big([\xi(\cdot)]^\alpha, [\zeta(\cdot)]^\alpha\Big) + cc_2\int_0^t d_H\Big([\xi(s)]^\alpha, [\zeta(s)]^\alpha\Big)ds \\
& \quad + cc_2\int_0^T d_H\Big([\xi(s)]^\alpha, [\zeta(s)]^\alpha\Big)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& d_\infty\Big((\varPhi\xi)(t), (\varPhi\zeta)(t)]\Big) \\
& \quad = \sup_{\alpha\in(0,1]} d_H\Big([(\varPhi\xi)(t)]^\alpha, [(\varPhi\zeta)(t)]^\alpha\Big) \\
& \quad \leq 2cc_1 d_\infty(\xi(\cdot),\zeta(\cdot)) + cc_2\int_0^t d_\infty(\xi(s),\zeta(s))ds + cc_2\int_0^T d_\infty(\xi(s),\zeta(s))ds.
\end{aligned}
$$

Hence

$$\begin{aligned}
H_1(\Phi\xi,\Phi\zeta) &= sup_{t\in J}d_\infty\Big((\Phi\xi)(t),(\Phi\zeta)(t)\Big)\\
&\le 2cc_1 sup_{\cdot\in J}d_\infty\Big(\xi(\cdot),\zeta(\cdot)\Big)+cc_2 sup_{t\in J}\int_0^t d_\infty\Big(\xi(s),\zeta(s)\Big)ds\\
&\quad +cc_2 sup_{t\in J}\int_0^T d_\infty\Big(\xi(s),\zeta(s)\Big)ds\\
&\le 2c(c_1+c_2T)H_1(\xi,\zeta).
\end{aligned}$$

Take T sufficiently small such that $T<\frac{1-2cc_1}{2cc_2}$, then Φ is a contraction mapping. By the Banach fixed point theorem, equation (2) is α-level exact controllable on J. □

4 Example

Consider the semilinear one dimensional heat equation on a connected domain $(0,1)$ for a material with memory, boundary conditions $x(t,0)=x(t,1)=0$, and with the initial and terminal conditions $x(0,z)-x(T,z)=\Psi(z)$, for $\Psi\in E_N$ (see [12]). Let $x(t,z)$ be the internal energy and $f(t,x(t,z))=\mathbf{2}tx(t,z)^2$ be the external heat.

Then the balance equation for this system becomes

$$\begin{aligned}
&x_t(t,z)=\mathbf{2}[x(t,z)-\int_0^t e^{-(t-s)}x(s,z)ds]_{zz}+u(t)+\mathbf{2}tx(t,z)^2,\\
&x(t,0)=x(t,1)=0,\\
&x(0,z)-x(T,z)=\mathbf{2}z^2\in E_N.
\end{aligned}$$

Let $G(t-s)=e^{-(t-s)}$, $\mathsf{A}=\mathbf{2}\frac{\partial^2}{\partial z^2}$, $B=1$, $f(t,x(t))=\mathbf{2}tx(t,z)^2$,

$$g(x(T,z))=x(T,z)=x(0,z)-\Psi(z)=\mathbf{2}z^2,$$

and $c_2=3T[|x_r^\alpha(t,z)+y_r^\alpha(t,z)|]>0$.

The α-level set of fuzzy number $\mathbf{2}$ is

$$[\mathbf{2}]^\alpha=[\alpha+1,3-\alpha] \text{ for all } \alpha\in[0,1].$$

Then the α-level set of $[f(t,x(t))]^\alpha$ is

$$\begin{aligned}
[\mathbf{2}tx(t,z)^2]^\alpha &= t[\mathbf{2}]^\alpha[x(t,z)^2]^\alpha\\
&= t[\alpha+1,3-\alpha][(x_q^\alpha(t,z))^2,(x_r^\alpha(t,z))^2]\\
&= t[(\alpha+1)(x_q^\alpha(t,z))^2,(3-\alpha)(x_r^\alpha(t,z))^2],
\end{aligned}$$

where $[x(t,z)]^\alpha = [x_q^\alpha(t,z), x_r^\alpha(t,z)]$. Further,

$$\begin{aligned}
&d_H([\Psi(x)]^\alpha, [\Psi(y)]^\alpha)\\
&\quad = d_H([(\alpha+1)(x_q^\alpha)^2, (3-\alpha)(x_r^\alpha)^2], [(\alpha+1)(y_q^\alpha)^2, (3-\alpha)(y_r^\alpha)^2])\\
&\quad \le \max\{(\alpha+1)|(x_q^\alpha)^2 - (y_q^\alpha)^2|, (3-\alpha)|(x_r^\alpha)^2 - (y_r^\alpha)^2|\}\\
&\quad \le (3-\alpha)\max\{|x_q^\alpha - y_q^\alpha||x_q^\alpha + y_q^\alpha|, |x_r^\alpha - y_r^\alpha||x_r^\alpha + y_r^\alpha|\}\\
&\quad \le 3|x_r^\alpha + y_r^\alpha|\max\{|x_q^\alpha - y_q^\alpha|, |x_r^\alpha - y_r^\alpha|\}\\
&\quad = c_r d_H([x]^\alpha, [y]^\alpha), \qquad c_r = 3|x_r^\alpha + y_r^\alpha|,
\end{aligned}$$

and g satisfies the inequality in hypothesis $(H1)$. Further,

$$\begin{aligned}
&d_H([f(t,x(t,z))]^\alpha, [f(t,y(t,z))]^\alpha)\\
&\quad = d_H(t[(\alpha+1)(x_q^\alpha(t,z))^2, (3-\alpha)(x_r^\alpha(t,z))^2],\\
&\qquad\quad t[(\alpha+1)(y_q^\alpha(t,z))^2, (3-\alpha)(y_r^\alpha(t,z))^2])\\
&\quad \le t\ \max\{(\alpha+1)|(x_q^\alpha(t,z))^2 - (y_q^\alpha(t,z))^2|,\\
&\qquad\quad (3-\alpha)|(x_r^\alpha(t,z))^2 - (y_r^\alpha(t,z))^2|\}\\
&\quad \le T(3-\alpha)\max\{|x_q^\alpha(t,z) - y_q^\alpha(t,z)||x_q^\alpha(t,z) + y_q^\alpha(t,z)|,\\
&\qquad\quad |x_r^\alpha(t,z) - y_r^\alpha(t,z)||x_r^\alpha(t,z) + y_r^\alpha(t,z)|\}\\
&\quad \le 3T|x_r^\alpha(t,z) + y_r^\alpha(t,z)| \times \max\{|x_q^\alpha(t,z) - y_q^\alpha(t,z)|, |x_r^\alpha(t,z) - y_r^\alpha(t,z)|\}\\
&\quad = c_2 d_H([x(t,z)]^\alpha, [y(t,z)]^\alpha).
\end{aligned}$$

This is an abstract formulation of equation (1). Further, the fuzzy number $S(t)$ exists (see [16]). Since f and g satisfy the global Lipschitz conditions, Theorem 3.3 implies that the fuzzy integrodifferential equation is α-level controllable on J.

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