

ADDITIVITY REGIONS FOR SOLUTIONS IN COOPERATIVE GAME THEORY

RODICA BRÂNZEI and STEF TIJS

Abstract. Superadditivity, subadditivity and additivity properties are considered for solutions and multisolutions of games in characteristic function form. Special attention is paid to the core and the τ -value and to cones of games on which these solutions are additive.

1 Introduction

A breakthrough in cooperative game theory was the paper of Lloyd Shapley (1953), where a solution concept for cooperative games was introduced i.e. a map assigning to each cooperative game a vector with as many coordinates as there are players. The solution which is nowadays known as the Shapley value has nice properties. One of the characterizing properties of the Shapley value in Shapley (1953) is the additivity property: the Shapley value of the sum of two games equals the sum of the Shapley values of the two original games. Many other solution concepts were introduced later. Many of them have not the additivity property on the whole linear space of games. The purpose of this paper is to discover and describe subcones of the game space on which a certain solution is additive.

Another important solution concept is the core (Gillies 1953), which assigns to each game a set of vectors. The core turns out to be a superadditive multi-solution: the core of the sum of two games is at least as large as the (Minkowski) sum of the cores of the two original games. In Drăgan, Potters and Tijds (1989) it is proved that on the cone of convex games the core is an additive correspondence. In this paper we will introduce other cones with this property such as the cones of simplex games, dual simplex games and general big boss games. Also other sub- and superadditive correspondences get attention. For a systematic study of cones in cooperative game theory we refer to Derks (1991).

The outline of the paper is as follows. Section 2 recalls some notions and facts of cooperative game theory which are useful later.

In Section 3 many solution concepts are investigated w.r.t. additivity properties. Special attention is given here to the τ -value. Section 4 deals with simplex games and dual simplex games and Section 5 with general big boss games. There is a summary of the main results in Section 6, where also some remarks on further research are made.

2 Preliminaries

In this section we recall some notions and facts of cooperative game theory which will be used in the following sections. An n -person cooperative game is a pair (N, v) , where

$N = \{1, 2, \dots, n\}$ is the *set of players* and $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function*, which assigns to each element of the family 2^N of subsets of N a real number and where $v(\emptyset) = 0$. The elements in 2^N are called *coalitions* and N is the *grand coalition*. The real number $v(S)$ is the *worth* or *value* of coalition S ; it is the amount which the coalition S can obtain when working together. The value $v(k)$ is called the *individual value* of player k and $i(v) = (v(1), v(2), \dots, v(n))^1$ is the *individual rational vector* of v . The real number $v^*(k) = v(N) - v(N \setminus \{k\})$ is called the *marginal contribution* of k to the grand coalition or also the *utopia payoff* for player k . It is the individual value of player k in the *dual game* $\langle N, v^* \rangle$, where $v^*(S) = v(N) - v(N \setminus S)$ for each $S \in 2^N$. If N is formed, the players can divide the amount $v(N)$. Player k will not be content with a non-individual rational payoff allocation, which gives less than $v(k)$ to him, because he can obtain $v(k)$ in staying alone. While $v(k)$ can be seen as a lower bound for the payoff of player k , the number $v^*(k)$ can be seen as an upper bound for the payoff. If a player k asks an amount z , which is larger than $v^*(k)$ in the grand coalition, then the other players in $N \setminus \{k\}$ can exclude him and $v(N \setminus \{k\}) > v(N) - z$ can be divided among the members of $N \setminus \{k\}$ if these players cooperate. The *utopia point* $u(v)$ of the game $\langle N, v \rangle$ is given by $u(v) = (v^*(1), v^*(2), \dots, v^*(n))$.

The *imputation set* $I(v)$ of $\langle N, v \rangle$ is defined by

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), x_i \geq v(i) \text{ for all } i \in N \right\}$$

and consists of all vectors in \mathbb{R}^n , satisfying the n *individual rationality conditions* and the *efficiency condition* $\sum_{i=1}^n x_i = v(N)$. Note that from the geometric point of view, the imputation

set $I(v)$ is equal to the intersection of the efficiency hyperplane $\left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(N) \right\}$ and the orthant $\{x \in \mathbb{R}^n \mid x \geq i(v)\}$ of individual rational payoff vectors.

The *dual imputation set* $I^*(v)$ of $\langle N, v \rangle$ is given by

$$I^*(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), x_i \leq v^*(i) \text{ for all } i \in N \right\},$$

and $I^*(v)$ is equal to the intersection of the efficiency hyperplane and the orthant $\{x \in \mathbb{R}^n \mid x \leq u(v)\}$ of subutopic vectors.

The set $C(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}$ is called the *core* of the game $\langle N, v \rangle$ and it consists of the efficient vectors x which are *split-off stable*, because no coalition S can obtain more than $\sum_{i \in S} x_i$ when splitting off. Related to the core is the *core cover* $CC(v) = I(v) \cap I^*(v)$ (cf. Tijs and Lipperts (1982)). The name is explained by the fact that for all games $CC(v) \supset C(v)$. All the introduced sets $I(v)$, $I^*(v)$, $C(v)$ and $CC(v)$

¹ We often write $v(i, j, \dots)$ instead of $v(\{i, j, \dots\})$.

may be empty. Necessary and sufficient conditions for the non-emptiness of these sets can be obtained with the aid of balancedness conditions, where a *balancedness condition* is an inequality of the form

$$v(N) \geq \sum_{S \in 2^N} \lambda_S v(S)$$

where $\lambda_S \geq 0$ for all $S \in 2^N$ and $\sum_{S: i \in S} \lambda_S = 1$ for all $i \in N$. So we have

- (i) $I(v) \neq \emptyset$ iff $v(N) \geq \sum_{i=1}^n v(i)$. The corresponding balancedness coordinates λ_S are given by $\lambda_S = 1$ if $|S| = 1$, and $\lambda_S = 0$ otherwise.
- (ii) $I^*(v) \neq \emptyset$ iff $v(N) \leq \sum_{i=1}^n v^*(i)$. Note that this inequality is equivalent with the balancedness condition $v(N) \geq \frac{1}{n-1} \sum_{i=1}^n v(N \setminus \{i\})$.
- (iii) $CC(v) \neq \emptyset$ iff $I(v) \cap I^*(v) \neq \emptyset$, which is equivalent to $\sum_{i \in N} v(i) \leq v(N) \leq \sum_{i \in N} v^*(i)$, and $v(i) \leq v^*(i)$ for all $i \in N$. Remark that $v(i) \leq v^*(i)$ corresponds to the balancedness condition $v(N) \geq v(N \setminus \{i\}) + v(i)$. Games with $CC(v) \neq \emptyset$ will be called *semi-balanced* and the cone of these games will be denoted by SB^N .
- (iv) $C(v) \neq \emptyset$ iff all balancedness conditions are satisfied. This is the famous result of Bondareva (1963) and Shapley (1967).

Given an ordering $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of the players in N , the σ -marginal vector $m^\sigma(v) \in \mathbb{R}^n$ of $\langle N, v \rangle$ is the vector which has for each $k \in N$ as $\sigma(k)$ -th coordinate the real number

$$v(\sigma(1), \sigma(2), \dots, \sigma(k)) - v(\sigma(1), \sigma(2), \dots, \sigma(k-1)).$$

Let us denote by $\Pi(N)$ the set of $n!$ orderings of N . Then the *Shapley value* $\phi(v)$ of $\langle N, v \rangle$ is equal to the average $\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v)$ of marginal vectors (cf. Shapley (1953)) and the

Weber set $W(v)$ (Weber (1988)) is the convex hull in \mathbb{R}^n of the set $\{m^\sigma(v) \mid \sigma \in \Pi(N)\}$ of marginal vectors.

For semi-balanced games $\langle N, v \rangle$, the σ -value is defined as the feasible compromise between $u(v)$ and $i(v)$. So $\sigma(v) = \alpha u(v) + (1 - \alpha)i(v)$, where the compromise factor α is the unique real number between 0 and 1, such that $\sum_{i=1}^n \sigma_i(v) = v(N)$. With SB_α^N we denote the cone of semi-balanced games with fixed *compromise factor* α .

The τ -value $\tau(v)$ (Tijs (1981)) for a quasi-balanced game is the feasible compromise between $u(v)$ and the *minimum right vector* $m(v)$, where for each $i \in N$

$$m_i(v) = \max_{S: i \in S} \left(v(S) - \sum_{j \in S \setminus \{i\}} v^*(j) \right),$$

and where a game is called *quasi-balanced* if

$$\sum_{i=1}^n m_i(v) \leq v(N) \leq \sum_{i=1}^n v^*(i), m_i(v) \leq v^*(i) \text{ for all } i \in \mathbb{N}.$$

So $\tau(v) = \alpha v + (1-\alpha)m(v)$, where α is the unique real number such that $\sum_{i=1}^n \tau_i(v) = v(N)$, which is called the τ -*compromise factor* in the following.

The cone of quasi-balanced games will be denoted by QB^N .

3 Superadditive, subadditive and additive solutions

Denote by G^N the set of all n -person games. G^N is a $(2^n - 1)$ -dimensional vector space with the usual operations of adding of functions, and scalar multiplication with real numbers, where each game $\langle N, v \rangle$ is identified with its characteristic function v . A *cone* K in G^N is a set of games with the property that $\alpha v + \beta w \in K$ if $v, w \in K$ and α, β are non-negative real numbers.

A map $\psi : K \rightarrow \mathbb{R}^n$ with as domain the cone $K \subset G^N$ is called *additive on K* if $\psi(v + w) = \psi(v) + \psi(w)$ for all $v, w \in K$.

Examples are given in the following.

Example 3.1. The Shapley value is additive on each cone K of G^N .

Example 3.2. We define for each $i \in N$ and each $v \in G^N$ the vectors $f^i(v)$ and $g^i(v)$ by

$$(f^i(v))_k = v(i) \text{ for each } k \in N \setminus \{i\} \text{ and } (f^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v(k)$$

$$(g^i(v))_k = v^*(i) \text{ for each } k \in N \setminus \{i\} \text{ and } (g^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v^*(k).$$

Then, obviously, f^i and g^i are additive maps from G^N into \mathbb{R}^n and also $\frac{1}{n} \sum_{i=1}^n f^i$ and $\frac{1}{n} \sum_{i=1}^n g^i$.

Example 3.3. Note that $I^N = \{v \in G^N \mid I(v) \neq \emptyset\}$ is a cone of games and also $I_*^N = \{v \in G^N \mid I^*(v) \neq \emptyset\}$. Now $CIS : I^N \rightarrow \mathbb{R}^n$ is defined by $CIS(v) = \frac{1}{n} \sum_{i=1}^n f^i(v)$ and CIS is,

in view of Example 3.2, an additive map, called the *center of the imputation set* solution. The name is explained by the fact that $f^1(v), f^2(v), \dots, f^n(v)$ are the extreme points of $I(v)$, which is an $(n - 1)$ -dimensional simplex if $v(N) > \sum_{i=1}^n v(i)$ and a one point set if

$$v(N) = \sum_{i=1}^n v(i).$$

On I_*^N the map $ENSR : I_*^N \rightarrow \mathbb{R}^n$ is additive, where $ENSR(v) = \frac{1}{n} \sum_{i=1}^n g^i(v)$. Here $ENSR$ stands for 'equal split of the non-separable rewards'.

Example 3.4. The map $\sigma : SB^N \rightarrow \mathbb{R}^n$ is not additive, but for each $\alpha \in [0, 1]$ the map $\sigma : SB_\alpha^N \rightarrow \mathbb{R}$ is additive which follows from the fact that for each $v \in SB_\alpha^N$

$$\sigma(v) = \alpha u(v) + (1 - \alpha)i(v), \text{ and } v \mapsto u(v) \text{ and } v \mapsto i(v)$$

are, obviously, additive.

Example 3.5. The τ -value is not additive on QB^N .

Let us analyse factors, which spoil the additivity of the τ -value on QB^N . On one hand, although the utopia map $v \mapsto u(v)$ is additive, the i -th coordinate of the minimum right map $v \mapsto m(v)$ is the maximum of additive maps of the form $v \mapsto v(S) - \sum_{j \in S \setminus \{i\}} v^*(j)$, and maximum taking destroys in general additivity. On the other hand, even if for a subcone K of QB^N the minimum right map is additive, then a second additivity destroying factor can be the fact that for different games in the cone the compromise factors may be different (see Example 3.6).

The above insight of factors which destroy additivity gives rise to the introduction of cones where the τ -value is additive.

Let $\alpha \in [0, 1]$, and for each $i \in N$ let B_i be a subset of N containing i .

Let $QB_{\alpha, B_1, B_2, \dots, B_n}^N$ be the cone of quasi-balanced games v with α as fixed uniform compromise factor and $B_i \in \arg \max_{S: i \in S} \left(v(S) - \sum_{j \in S \setminus \{i\}} v^*(j) \right)$.

Then, obviously, the minimum right map is additive on this cone and also the τ -value, because $\tau(v) = \alpha u(v) + (1 - \alpha)m(v)$ for all $v \in QB_{\alpha, B_1, B_2, \dots, B_n}^N$. So we obtain

Theorem 3.1. *Let $\alpha, B_1, B_2, \dots, B_n$ be as above. Then $\tau : QB_{\alpha, B_1, B_2, \dots, B_n}^N \rightarrow \mathbb{R}^n$ is an additive map.*

Example 3.6. Recall that a game $v \in G^N$ is said to be *convex* (Shapley (1971)) if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N$. For convex games the minimum right map is additive because $m(v) = (v(1), v(2), \dots, v(n))$ for each $v \in CONV^N$. But $\tau : CONV^N \rightarrow \mathbb{R}^n$ is not additive.

An explanation of the result in Example 3.6 is that there is no uniform compromise factor for $CONV^N$. E.g. each unanimity game $u_S : 2^N \rightarrow \mathbb{R}$ such that $u_S(T) = 1$ if $S \subset T$, and $u_S(T) = 0$ otherwise, is a convex game. But for $|S| \geq 2$ the compromise factor $\alpha(u_S) = |S|^{-1}$, depends on the size of $S \in 2^N \setminus \{\emptyset\}$, which follows from $u(u_S) = u_S(N)e^S$, $m(u_S) = 0$, so $\tau(u_S) = |S|^{-1}u(u_S) + (1 - |S|^{-1})0$. Here e^S is the characteristic vector for S , with $(e^S)_i = 1$ if $i \in S$, and $(e^S)_i = 0$ otherwise. So, let us look at the subcones K^r ($r = 1, 2, \dots, n$) of $CONV^N$ with uniform compromise factor r^{-1} , given by

$$K^r = \text{cone}\{u_S \mid S \in 2^N, |S| = r\},$$

the cone generated by the unanimity games based on a set S of size r .

Then we have

Theorem 3.2. *The τ -value $\tau : K^r \rightarrow \mathbb{R}^n$ is an additive map.*

The cones K^2 and K^3 arose in connection with telecom problems (Nouweland et al. (1996)). In the mentioned paper it is proved that for all $r \in \{1, \dots, n\}$ the τ -value and the Shapley value coincide on K^r . On K^1 and K^2 also the nucleolus coincides with the two other values.

Now we turn to multisolutions, which are correspondences $D : K \rightarrow \mathbb{R}^n$ from a cone $K \subset G^N$ into the family of subsets of \mathbb{R}^n . Such a correspondence is called *superadditive* on K if for all $v, w \in K$ we have $D(v+w) \supset D(v) + D(w)$, and *subadditive* if the reverse inclusion holds. A multifunction $D : K \rightarrow \mathbb{R}^n$ is called additive if the function is superadditive and subadditive or, equivalently, if $D(v+w) = D(v) + D(w)$ for all $v, w \in K$.

Example 3.7. $C : G^N \rightarrow \mathbb{R}^n$ is a superadditive correspondence and $W : G^N \rightarrow \mathbb{R}^n$ is subadditive (see e.g. Drăgan, Potters and Tijs (1989)).

Example 3.8. Using the results in Example 3.7 and the fact that for the cone $CONV^N$ of convex games the core and the Weber set coincide, Drăgan et al. (1989) proved

Theorem 3.3. *$C : CONV^N \rightarrow \mathbb{R}^n$ is additive.*

Example 3.9. The cone of exact games EX^N was introduced by Schmeidler (1972) and contains the cone of convex games. Recall that a game $v \in G^N$ is *exact* if for each $S \in 2^N \setminus \{\emptyset\}$, there is an $x \in C(v)$ such that $x(S) = v(S)$. The core correspondence on EX^N is, of course, superadditive but not additive which we learn from the symmetric 4-person exact games $v, w \in EX^N$, given by

$$\begin{aligned} v(S) &= 0, 0, 2, 3, 6 \text{ if } |S| = 0, 1, 2, 3, 4, \text{ respectively, and} \\ w(S) &= 0, 0, 2, 5, 8 \text{ if } |S| = 0, 1, 2, 3, 4, \text{ respectively.} \end{aligned}$$

Then $z = (0, 4, 4, 6) \in C(v+w)$, and $x = (0, 2, 2, 2)$ is the unique element in $C(v)$ with as first coordinate 0, which implies that there is no $y \in C(w)$ with $z = x + y$. So $C(v+w)$ is really larger than $C(v) + C(w)$.

Example 3.10. The maps $I : G^N \rightarrow \mathbb{R}^n$, $I^* : G^N \rightarrow \mathbb{R}^n$, assigning respectively to a game the imputation set and the dual imputation set are superadditive. Since the core cover $CC = I \cap I^*$ is the intersection of two superadditive correspondences, it is also superadditive. The core cover is not additive. For I and I^* we can say more as the following theorem shows.

Theorem 3.4. *$I : I^N \rightarrow \mathbb{R}^n$, $I^* : I_*^N \rightarrow \mathbb{R}^n$ are additive correspondences.*

PROOF. We noted already that the correspondences are superadditive on G^N , so certainly on I^N and I_*^N , respectively.

(i) To prove the additivity of $I : I^N \rightarrow \mathbb{R}^n$ we have to show that for $v, w \in I^N$ each $z \in I(v+w)$ can be written as $z = x + y$ with $x \in I(v)$ and $y \in I(w)$. Note that z is a convex combination of the extreme points $f^1(v+w), f^2(v+w), \dots, f^n(v+w)$ of the simplex $I(v+w)$, so there are non-negative real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 1$, such that

$$z = \sum_{i=1}^n \alpha_i f^i(v+w).$$

Because $f^i(v+w) = f^i(v) + f^i(w)$ for each $i \in N$, we obtain the result $z = x + y$ with $x = \sum_{i=1}^n \alpha_i f^i(v) \in I(v)$ and $y = \sum_{i=1}^n \alpha_i f^i(w) \in I(w)$. Hence, $I : I^N \rightarrow \mathbb{R}^n$ is additive.

(ii) The proof of the additivity of $I^* : I_*^N \rightarrow \mathbb{R}^n$ runs in a similar way since each $z \in I^*(v+w)$ is a convex combination of the extreme points $g^1(v+w), g^2(v+w), \dots, g^n(v+w)$ of $I^*(v+w)$, and $g^i(v+w) = g^i(v) + g^i(w)$ for all $i \in N$. \square

4 Simplex games and dual simplex games

Let us call a game $\langle N, v \rangle$ a *simplex game* if the core $C(v)$ of the game v is non-empty and equals the simplex $I(v)$. A game $\langle N, v \rangle$ is called a *dual simplex game* if $\emptyset \neq C(v) = I^*(v)$. Let us denote the set of simplex games by SI^N and the set of dual simplex games by SI_*^N . It will turn out that these sets are cones on which the core is an additive correspondence.

Theorem 4.1.

- (i) $SI^N = \left\{ v \in G^N \mid v(S) \leq \sum_{i \in S} v(i) \text{ for all } S \neq N, \sum_{i \in N} v(i) \leq v(N) \right\}$
- (ii) SI^N is a cone
- (iii) $C : SI^N \rightarrow \mathbb{R}^n$ is additive.

PROOF. (i.a) Suppose $v \in SI^N$. Then $I(v) \neq \emptyset$, so $v(N) \geq \sum_{i \in N} v(i)$. Let $S \in 2^N, S \neq N$. Take $k \in N \setminus S$. Since $f^k(v) \in I(v) = C(v)$, we obtain $\sum_{i \in S} v(i) = \sum_{i \in S} (f^k(v))_i \geq v(S)$. So we have proved that SI^N is a subset of the set of the right side of the equality sign in (i).

(i.b) For the converse inclusion, suppose that $v \in G^N$ is such that $v(S) \leq \sum_{i \in S} v(i)$ for all $S \neq N$ and $\sum_{i \in N} v(i) \leq v(N)$. Then the last inequality implies that $I(v) \neq \emptyset$. Of course, $C(v) \subset I(v)$, but the reverse also holds because for each $x \in I(v)$ we have $\sum_{i=1}^n x_i = v(N)$ and for all $S \neq N : \sum_{i \in S} x_i \geq \sum_{i \in S} v(i) \geq v(S)$. So $\emptyset \neq I(v) = C(v), v \in SI^N$. This finishes the proof of (i).

(ii) From the characterization of SI^N in terms of linear inequalities in (i) it follows directly that SI^N is a cone.

(iii) The additivity of $C : SI^N \rightarrow \mathbb{R}^n$ follows directly from (ii) and Theorem 3.4. \square

Note that (i) of Theorem 4.1 tells that in simplex games it does not pay to form coalitions unequal to the grand coalition.

Let us look at the subcone $SUSI^N$ of SI^N of superadditive simplex games. Recall that an n -person game is called a superadditive game if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$.

Theorem 4.2.

(i) For each $v \in SUSI^N$:

$$CIS(v) = ENSR(v) = \tau(v), \quad CC(v) = C(v)$$

(ii) $\tau : SUSI^N \rightarrow \mathbb{R}^n$ is additive

(iii) $CC : SUSI^N \rightarrow \mathbb{R}^n$ is additive.

PROOF. (i) For $v \in SUSI^N$ and $i \in N$: $v^*(i) = v(N) - v(N \setminus \{i\}) = v(N) - \sum_{k \in N \setminus \{i\}} v(k)$. So

$$ENSR_i(v) = v^*(i) - \frac{1}{n} \left(\sum_{k=1}^n v^*(k) - v(N) \right) = v^*(i) - \frac{n-1}{n} \left(v(N) - \sum_{k \in N} v(k) \right) = v(i) + \frac{1}{n} \left(v(N) - \sum_{k \in N} v(k) \right) = CIS_i(v).$$

Further, $m_i(v) = v(i)$ for each $i \in N$. Then

$$\tau_i(v) = \frac{1}{n} v^*(i) + \frac{n-1}{n} m_i(v) = v(i) + \frac{1}{n} \left(v(N) - \sum_{k \in N} v(k) \right) = CIS(v)$$

$$\begin{aligned} CC(v) &= \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(N), v(i) \leq x_i \leq v^*(i) \text{ for each } i \in N \right\} \\ &= \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(N), v(i) \leq x_i \leq v(N) - \sum_{k \in N \setminus \{i\}} v(k) \right\} = \\ &= I(v) = C(v). \end{aligned}$$

(ii) Since $CIS : I^N \rightarrow \mathbb{R}^n$ is additive also $CIS : SUSI^N \rightarrow \mathbb{R}^n$ is additive and $\tau = CIS$ on $SUSI^N$ by (i).

(iii) $CC : SUSI^N \rightarrow \mathbb{R}^n$ is additive since $CC = C$ on $SUSI^N$ and C is additive on $SUSI^N$ by Theorem 4.1. \square

The next theorem can be seen as a dual of Theorem 4.1.

Theorem 4.3.

$$(i) \quad SI_*^N = \left\{ v \in G^N \mid v^*(S) \geq \sum_{i \in S} v^*(i) \text{ for all } S \neq N, \sum_{i \in N} v^*(i) \geq v^*(N) \right\}$$

(ii) SI_*^N is a cone

(iii) $C : SI_*^N \rightarrow \mathbb{R}^n$ is an additive correspondence.

PROOF. (i.a) First we prove that SI_*^N is a subset of the set on the right side of the equality in (i). In (i.b) we prove the converse inclusion. Let $v \in SI_*^N(v)$. Then $\emptyset \neq C(v) = I^*(v)$ implies that $\sum_{i \in N} v^*(i) \geq v(N) = v^*(N)$. Let $S \in 2^N \setminus \{N\}$. Take $i \in N \setminus S$. Then $g^i(v) \in I^*(v) =$

$C(v)$ implies that $v(N \setminus S) \leq \sum_{i \in N \setminus S} g^i(v) = \sum_{k \in N \setminus S \cup \{i\}} v^*(k) + \left(v(N) - \sum_{k \in N \setminus \{i\}} v^*(k) \right) = v(N) - \sum_{k \in S} v^*(k)$. So $v^*(S) = v(N) - v(N \setminus S) \geq \sum_{k \in S} v^*(k)$.

(i.b) Suppose that $v^*(S) \geq \sum_{i \in S} v^*(i)$ for all $S \neq N$ and $\sum_{i \in N} v^*(i) \geq v^*(N)$. Then the last inequality implies that $I^*(v) \neq \emptyset$. Further $C(v) \subset I^*(v)$. For the converse we prove that $g^i(v) \in C(v)$ for each $i \in N$. Clearly, $\sum_{k=1}^n (g^i(v))_k = v(N)$. Since $g^i(v) \in I^*(v)$ we have for each $S \in 2^N, S \neq N$: $\sum_{k \in N \setminus S} (g^i(v))_k \leq \sum_{k \in N \setminus S} v^*(k) \leq v^*(N \setminus S)$. So $\sum_{k \in S} (g^i(v))_k = v(N) - \sum_{k \in N \setminus S} (g^i(v))_k \geq v(N) - v^*(N \setminus S) = v(S)$. Hence $g^i(v) \in C(v)$.

(ii) follows from (i), and (iii) follows from the additivity of I^* on SI_*^N .

In Driessen and Tijs (1983) and Driessen (1988) the cone $CONV_1^N$ of 1-convex games was studied. $CONV_1^N$ was defined as $\{v \in G^N \mid 0 \leq g^v(N) \leq g^v(S) \text{ for each } S \in 2^N\}$, where $g^v(S) := \sum_{i \in S} v^*(i) - v(S)$ is the gap of coalition S in the game v , or the gap between the sum of the utopia payoffs of S and the reality that only $v(S)$ can be obtained by S in cooperation. For these games $v \in CONV_1^N$ it was proved that $\tau(v) = ENSR(v)$. Note that $g^v(N) \geq 0$ is equivalent with $\sum_{i \in N} v^*(i) \geq v(N)$ and that for $S \neq N$: $g^v(N \setminus S) \geq g^v(N)$ is equivalent to $\sum_{i \in N \setminus S} v^*(i) - v(N \setminus S) \geq \sum_{i \in N} v^*(i) - v(N)$ and so also to $\sum_{i \in S} v^*(i) \leq v(N) - v(N \setminus S) = v^*(S)$. This implies that $CONV_1^N = SI_*^N$. So we have in view of the above

Theorem 4.4.

- (i) $\tau(v) = ENSR(v)$ for all $v \in SI_*^N$
- (ii) $\tau : SI_*^N \rightarrow \mathbb{R}^n$ is additive.

5 General big boss games

In this section we look at n -person games $\langle N, v \rangle$, where one player – namely player n – has a special role. To be precise, we consider the class of games $GBB^N = \{v \in G^N \mid f^n(v) \in C(v), g^n(v) \in C(v)\}$ and call its elements *general big boss games* for reasons which become clear soon.

Theorem 5.1. *Let $v \in G^N$. The following assertions are equivalent*

- (i) $v \in GBB^N$

(ii) v satisfies the following three properties:

$$v(i) \leq v^*(i) \text{ for all } i \in N, \sum_{i \in N} v(i) \leq v(N) \leq \sum_{i \in N} v^*(i) \text{ [core cover non-emptiness]}$$

$$v(S) \leq \sum_{i \in S} v(i) \text{ for all } S \in 2^N \text{ with } n \notin S \quad \text{[general big boss property]}$$

$$v^*(N \setminus S) \geq \sum_{i \in N \setminus S} v^*(i) \text{ for all } S \in 2^N \text{ with } n \in S \text{ [union property]}$$

(iii) $\phi \neq C(v)$

$$= \left\{ x \in \mathbb{R}^n \mid v(i) \leq x_i \leq v^*(i) \text{ for all } i \in N \setminus \{n\}, \sum_{i=1}^n x_i = v(N) \right\}.$$

PROOF. We prove respectively that (i) implies (ii), (ii) implies (iii), and (iii) implies (i).

(a) Suppose $v \in GBB^N$. Then $C(v)$ is non-empty, which implies that $CC(v)$ is non-empty since $C(v) \subset CC(v)$. The fact that $f^n(v) \in C(v)$ implies for S not containing n that $v(S) \leq \sum_{i \in S} (f^n(v))_i = \sum_{i \in S} v(i)$. So the general big boss property holds. The property $g^n(v) \in C(v)$ implies the union property, since for S with $n \in S$ we have

$$\begin{aligned} v(S) &\leq \sum_{i \in S} (g^n(v))_i = \left(v(N) - \sum_{i \in N \setminus \{n\}} v^*(i) \right) + \sum_{i \in S \setminus \{n\}} v^*(i), \\ \text{or } v^*(N \setminus S) &= v(N) - v(S) \geq \sum_{i \in N \setminus S} v^*(i) \end{aligned}$$

(b) Suppose v satisfies the three properties in (ii). We have to show that (iii) holds. It is sufficient to show that $\phi \neq P(v) = C(v)$ where

$$P(v) = \left\{ x \in \mathbb{R}^n \mid v(i) \leq x_i \leq v^*(i) \text{ for all } i \in N \setminus \{n\}, \sum_{i=1}^n x_i = v(N) \right\}.$$

Note that $C(v) \subset CC(v) \subset P(v)$. Because of the property in (ii) that the core cover is not empty we obtain that $P(v) \neq \emptyset$. Further $C(v) \subset P(v)$. We have finished the proof if we show that $x \in P(v)$ satisfies the split-off stability conditions: $\sum_{i \in S} x_i \geq v(S)$ for each $S \in N$.

Take first an $S \in 2^N$ with $n \notin S$. Then $\sum_{i \in S} x_i \geq \sum_{i \in S} v(i) \geq v(S)$ by the general big boss property. Take finally an $S \in 2^N$ with $n \in S$. Then $\sum_{i \in N \setminus S} x_i \leq \sum_{i \in N \setminus S} v^*(i) \leq v^*(N \setminus S)$ by

the union property. So $\sum_{i \in S} x_i = \sum_{i=1}^n x_i - \sum_{i \in N \setminus S} x_i \geq v(N) - v^*(N \setminus S) = v(S)$.

(c) That (iii) implies (i) is obvious, because $(f^n(v))_i = v(i)$ and $(g^n(v))_i = v^*(i)$ for each $i \in N \setminus \{n\}$; further $\sum_{i=1}^n (f^n(v))_i = v(N) = \sum_{i=1}^n (g^n(v))_i$. \square

Remark. For $v \in GBB^N$, player n is called the big boss, because of the general big boss property, which tells that a coalition not containing n cannot profit from cooperation. Such a coalition is called a *union*. The union property tells that for a union $N \setminus S$ (so $n \in S$) its marginal contribution $v(N) - v(S)$ to the grand coalition is at least as large as the sum of the individual marginal contributions $v^*(i)$ of its members.

Remark. For a general big boss game with three players, the core is a parallelogram in the efficiency plane. For $v \in GBB^n$ the core is the intersection of $n - 1$ strips S_1, S_2, \dots, S_{n-1} and the efficiency hyperplane $\left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(N) \right\}$. Here S_i is the region $\{x \in \mathbb{R}^n \mid v(i) \leq x_i \leq v^*(i)\}$ between the two parallel hyperplanes $\{x \in \mathbb{R}^n \mid x_i = v(i)\}$ and $\{x \in \mathbb{R}^n \mid x_i = v^*(i)\}$. We will denote the average $\frac{1}{2}(f^n(v) + g^n(v))$ by $\beta(v)$, because this point is the barycenter of the core.

Remark. In Muto et al. (1989) the cone BB^N of big boss games was introduced and many economic situations giving rise to such games were discussed. Here we call a game v a *big boss game with player n as big boss* if (i) $v \in GBB^N$, (ii) $v(S) = \sum_{i \in S} v(i)$ for all $S \in 2^N$ with $n \in S$. In Muto et al. (1989) only 0-normalized big boss games were considered, i.e. games with $v(i) = 0$ for each $i \in N$.

For big boss games it was proved that the τ -value is in the center of the core and coincides with the nucleolus.

We collect some interesting facts for general big boss games in the next theorem.

Theorem 5.2.

- (i) GBB^N is a cone,
- (ii) $CC(v) = C(v)$ for all $v \in GBB^N$,
- (iii) $C : GBB^N \rightarrow \mathbb{R}^n$ is an additive correspondence,
- (iv) $\beta : GBB^N \rightarrow \mathbb{R}^n$ is an additive map,
- (v) $\tau(v) = \beta(v)$ for $v \in GGB^N$ iff $v(N \setminus \{n\}) = \sum_{i=1}^{n-1} v(i)$,
- (vi) For $v \in BB^N : \tau(v) = \beta(v)$ and τ is additive on BB^N .

PROOF. (i) That GBB^N is a cone follows from characterization (ii) of general big boss games in Theorem 5.1.

(ii) That $CC(v) = C(v)$ for $v \in GBB^N$ follows from part (b) of the proof of Theorem 5.1.

(iii) Let $v, w \in GBB^N$ and let $z \in C(v + w)$. We have to prove that there are $x \in C(v)$, $y \in C(w)$ such that $z = x + y$. In view of characterization (iii) of general big boss games, we can find $\alpha_1, \alpha_2, \dots, \alpha_{n-1} \in [0, 1]$ such that $z_i = \alpha_i(v + w)(i) + (1 - \alpha_i)(v + w)^*(i)$ for each $i \in N \setminus \{n\}$. Take $x \in C(v)$ and $y \in C(w)$ such that $x_i = \alpha_i v(i) + (1 - \alpha_i)v^*(i)$ and $y_i = \alpha_i w(i) + (1 - \alpha_i)w^*(i)$ for all $i \in N \setminus \{n\}$. Then $z = x + y$.

(iv) That β is additive follows from the additivity of f^n and g^n .

(v) Note that in view of (ii) of Theorem 5.1, for $v \in GBB^N$

$$m(v) = \left(v(1), v(2), \dots, v(n-1), v(N) - \sum_{i=1}^{n-1} v^*(i) \right)$$

$$u(v) = (v^*(1), \dots, v^*(n-1), v(N) - v(N \setminus \{n\})).$$

So $\tau(v) = \alpha u(v) + (1 - \alpha)m(v)$ for some $\alpha \in [0, 1]$. Then $\tau(v) = \beta(v)$ iff $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ iff $v(N \setminus \{n\}) = \sum_{i=1}^{n-1} v(i)$ because

$$\sum_{i=1}^n u_i(v) - v(N) = \sum_{i=1}^{n-1} v^*(i) - v(N \setminus \{n\}),$$

$$v(N) - \sum_{i=1}^n m_i(v) = \sum_{i=1}^{n-1} v^*(i) - \sum_{i=1}^{n-1} v(i).$$

(vi) For $v \in BB^N$ we have $v(S) = \sum_{i \in S} v(i)$ for each S with $n \notin S$. So by (v), $\tau(v) = \beta(v)$, and by (iv) $\tau : BB^N \rightarrow \mathbb{R}^n$ is an additive map. \square

6 Summary and concluding remarks

Starting points of the paper were the observations that the Shapley value is additive on all cones of games and that the core is additive on the cone of convex games. Although most of the other interesting solutions do not have this global additivity property it turned out that many of them have the additivity property on interesting cones of games such as the *CIS*-solution and the *ENSR*-solution. By analysing two additivity destroying factors for the τ -value, we obtained a collection of cones (Theorem 3.1), where the τ -value is additive. Similarly, cones based on uniform compromise factors were found for the σ -value (Example 3.4). For multi-solutions such as the core and the core cover also new regions of additivity were found. The core is additive not only on the cone of convex games but also e.g. on the cones of simplex games, dual simplex games and general big boss games and so is the τ -value on subcones of these cones. A negative result was found for the cone of exact games when the core is not additive (Example 3.9).

We conclude with some remarks on further research. A systematic approach in looking for additivity regions of the core will yield new interesting cones of games. Few attention is paid to additivity regions of the nucleolus (Schmeidler (1969)) although we know that the nucleolus coincides with the τ -value for the cones of big boss games and of dual simplex games, so on these cones the nucleolus is also additive. Exploration of other cones of additivity for the nucleolus and other solution concepts may also enrich cooperative game theory.

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