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# INVARIANT SOLUTIONS OF RADIATIVE MAGNETOHYDRODYNAMICS

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**Abstract.** In this paper we give two types of solutions for one-dimensional radiative magnetohydrodynamics which are invariant to the transformations of the admitted group of these equations. We prove that in both cases the problem can be reduced to the integration of normal systems of ordinary differential equations.

## 1 Introduction

The idea of using continuous groups of transformations to study the differential equations belongs to S. Lie. It was afterwards resumed by G. Birkhoff in the study of the hydrodynamics equations, as well as by L. V. Ovsianicov [1], [2], [3] and Gh. Gheorghiev [4], [5]. Knowing the most wide admitted group by a differential system, using group properties, we can find out particular solutions and can research the structure of its solutions set.

The problem is analyzed in the frame of Lie's classical theory, so that all the groups and differential equations are assumed analytical.

Let $G_r$ be a Lie local group of punctual transformations of the Euclidean space $\mathbf{R}^N(x,u)$, where $N=n+m$, $x=(x^1,x^2,\ldots,x^n)$, $u=(u^1,u^2,\ldots,u^m)$, and let $L_r$ be the Lie algebra with

$$X_\alpha=\xi_\alpha^i(x,u)\frac{\partial}{\partial x^i}+\eta_\alpha^k(x,u)\frac{\partial}{\partial u^k},\ \ \alpha=1,\ldots,r,$$

as a basis of infinitesimal operators, where $\xi_\alpha^i$ and $\eta_\alpha^k$ are functions of class $C^\infty(\mathbf{R}^N)$. Let $\mathbf{R}^{\widetilde{N}}(x,u,p)$, where $p_i^k=\dfrac{\partial u^k}{\partial x^i}$, be differential extension of $\mathbf{R}^N$. The action of $G_r$ on $\mathbf{R}^N$ can be canonical extended to an action of extended group $\widetilde{G}_r$ on $\mathbf{R}^{\widetilde{N}}$. The Lie algebra of $\widetilde{G}_r$ is isomorphic with $L_r$ and admits the basis

$$\widetilde{X}_\alpha=X_\alpha+\zeta_{\alpha i}^k\frac{\partial}{\partial p_i^k},\ \ \alpha=1,\ldots,r,$$

with

$$\zeta_{\alpha i}^k=D_i(\eta_\alpha^k)-p_j^kD_i(\xi_\alpha^j),\ \ D_i=\frac{\partial}{\partial x^i}+p_i^k\frac{\partial}{\partial u^k}.$$

Consider the following system of first-order differential equations:

$$F^a(x,u,p)=0,\ \ a=1,\ldots,A, \tag{1.1}$$

where $u^k$ are unknown functions of $x^i$. The equations of (1.1) can be taken as the equations of a variety in extended space $\mathbf{R}^{\widetilde{N}}$. Denote this variety with $S$.

We say that the system (1.1) admits the group $G_r$ if the correspondent variety $S$ is an invariant differential variety of the group $G_r$.

The system (1.1) admits the group $G_r$, with the operators of Lie algebra $X_\alpha$, if and only if,

$$\widetilde{X}_\alpha F^a (x, u, p)\Big|_S = 0, \tag{1.2}$$

for every $(x, u, p) \in S$.

The equations (1.2) form a system of differential equations for the coordinates $\xi$ and $\eta$ of the operators $X_\alpha$.

Let $H$ be a subgroup of the group $G_r$. The solution $\Phi$ of the system (1.1) is termed invariant solution to the subgroup $H$ if $\Phi$ is an invariant variety of the subgroup $H$ admitted by the system.

If $R$ is the generic rank of the coordinate matrix of the operators that form a basis of the subgroup $H$, then the number $\rho = \dim \Phi - R$ is termed the rank of the invariant solution $\Phi$.

Invariant solutions exist if the subgroup $H$ is intransitive, i.e. it has invariants, therefore $R < \widetilde{N}$. Let

$$I^\tau = I^\tau (t, x, u), \quad \tau = 1, \ldots, \widetilde{N} - R,$$

be a complete system of independent invariants of the subgroup $H$. There are invariant solutions if

$$\operatorname{Rank}\left(\frac{\partial I^\tau (t, x, u)}{\partial u^b}\right) = N. \tag{1.3}$$

## 2 Invariance groups of radiative magnetohydrodynamics equations

The equations of radiative magnetohydrodynamics for an ideal perfectly conducting gas for one-dimensional flow are given in [7]:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho}\frac{\partial p^m}{\partial x} + \frac{B_2}{\mu\rho}\frac{\partial B_2}{\partial x} + \frac{B_3}{\mu\rho}\frac{\partial B_3}{\partial x} = \tag{2.1}$$

$$= \frac{1}{\rho c^2}\left[\alpha\left(S_1^r - \frac{4}{3}E^r u_1\right) + P^r u_1\right],$$

$$\frac{\partial u_j}{\partial t} + u_1 \frac{\partial u_j}{\partial x} - \frac{B}{\mu\rho}\frac{\partial B_j}{\partial x} = \frac{1}{\rho c^2}\left[\alpha\left(S_j^r - \frac{4}{3}E^r u_j\right) + P^r u_j\right], \quad j = 1, 2, \tag{2.2}$$

$$\frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_1}{\partial x} = 0, \tag{2.3}$$

$$\frac{\partial p^m}{\partial t} + u_1 \frac{\partial p^m}{\partial x} + \left[\gamma p^m + \frac{4}{3}(\gamma - 1) E^r\right]\frac{\partial u_1}{\partial x} + (\gamma - 1) u_1 \frac{\partial E^r}{\partial x} = \tag{2.4}$$

$$= (\gamma - 1) P^r,$$

$$\frac{B_j}{\partial t} + u_1 \frac{\partial B_j}{\partial x} + B_j \frac{\partial u_1}{\partial x} - B \frac{\partial u_j}{\partial x} = 0, \quad j = 1, 2, \tag{2.5}$$

$$\frac{\partial E^r}{\partial t} + \frac{\partial S_1^r}{\partial x} = -P^r, \tag{2.6}$$

$$\frac{\partial S_1^r}{\partial t} + \frac{c^2}{3} \frac{\partial E^r}{\partial x} = -\alpha \left( S_1^r - \frac{4}{3} E^r u_1 \right) - P^r u_1, \tag{2.7}$$

$$\frac{\partial S_j^r}{\partial t} = -\alpha \left( S_j^r - \frac{4}{3} E^r u_j \right) - P^r u_j, \quad j = 1, 2, \tag{2.8}$$

where $u_i(t,x)$ is the fluid velocity, $\rho(t,x)$ is the mass density of fluid, $p^m(t,x)$ is the fluid pressure, $B_i(t,x)$ is the magnetic induction with

$$B_1(t,x) = B = \text{const},$$

$E^r(t,x)$ is the radiation energy density, $S_i(t,x)$, is the radiative flux,

$$P^r = \alpha \left( E^r - aT^4 \right),$$

$i = 1, 2, 3$, and $\gamma$, $\alpha$ and $a$ are constants.

Denoting by $u^a$, $a = \overline{1, 11}$, the 11 unknown functions from the system (2.1)-(2.8), $x^0 = t$, $x^1 = x$, $\dfrac{\partial u^a}{\partial x^0} = -p_0^a$, $\dfrac{\partial u^a}{\partial x^1} = p^a$, the previous system takes the form:

$$p_0^a = F_{p^b}^a p^b + F_0^a, \quad a, b = \overline{1, 11}. \tag{2.9}$$

We apply here the procedure from [6] for the determination of the invariance groups of the system (2.9) by means of Lie algebra of its infinitesimal operators. Let

$$X = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^a \frac{\partial}{\partial u^a} = \xi^\alpha p_\alpha + \eta^a q_a,$$

be such an operator. Then, a necessary and sufficient condition that the system (2.9) be invariant with respect to the operator $X$ is that the functions $\xi^\alpha(x,u)$ and $\eta^a(x,u)$ be a solution of the equations:

$$\xi_{u^d}^0 F_{p^{(b}}^a F_{p^{c)}}^d + \xi_{u^d}^1 \delta_{(b}^a F_{p^{c)}}^d - F_{p^d}^a F_{p^{(b}}^d \xi_{u^{c)}}^0 - F_{p^{(b}}^a \xi_{u^{c)}}^1 = 0, \tag{2.10}$$

$$\begin{gathered} \eta_{u^c}^a F_{p^b}^c - \delta_b^a \left( \xi_{x^0}^1 + \xi_{u^c}^1 F_0^c \right) - F_{p^b}^a \left( \xi_{x^0}^0 + \xi_{u^c}^0 F_0^c \right) - F_0^a \xi_{u^c}^0 F_{p^b}^c - \xi^\alpha F_{p^b, x^\alpha}^a - \\ - \eta^c F_{p^b, u^c}^a - F_{p^c}^a \eta_{u^b}^c + F_{p^c}^a F_{p^b}^c \xi_{x^1}^0 + F_{p^c}^a F_0^c \xi_{u^b}^0 + F_{p^b}^a \xi_{x^1}^1 = 0. \end{gathered} \tag{2.11}$$

$$\eta_{x^0}^a + \eta_{u^c}^a F_0^c - F_0^a \xi_{u^c}^0 F_0^c - \xi^\alpha F_{0, x^\alpha}^a - \eta^c F_{0, u^c}^a - \eta_{x^1}^b F_{p^b}^a + F_{p^b}^a F_0^b \xi_{x^1}^0 = 0, \tag{2.12}$$

where the subscripts $x^\alpha$, $u^a$ from $\xi$, $\eta$ and after the comma from $F_0^a$ and $F_{p^b}^a$ denote the partial derivatives, and $(b, c)$ is the symmetrization of the pair $b, c$.

The Lie groups associated with these Lie algebras are given in [7].

## 3 Invariant solutions

Consider the system (2.9) in which we neglect the terms from the right-hand members that depend on velocity. The local Lie group $G_5$ of the punctual transformations of the Euclidean space $\mathbf{R}^{13}(t,x,u^a)$ — admitted by this system — has Lie algebra determined by the basis of the infinitesimal operators:

$$X_1=\frac{\partial}{\partial t},\ \ X_2=\frac{\partial}{\partial x},\ \ X_3=\frac{\partial}{\partial u_2},\ \ X_4=\frac{\partial}{\partial u_3},$$

$$X_5=u_3\frac{\partial}{\partial u_2}-u_2\frac{\partial}{\partial u_3}+B_3\frac{\partial}{\partial B_2}-B_2\frac{\partial}{\partial B_3}+S_3^r\frac{\partial}{\partial S_2^r}-S_2^r\frac{\partial}{\partial S_3^r}.$$

In what follows we deal with two types of invariant solutions $\Phi$ with $\dim\Phi=2$, $\rho=1$, and then $R=1$, i.e. $\dim H=1$.

**1**$^0$. The subgroup $H$ admitted by the Lie algebra determined by the infinitesimal operator

$$X=\frac{1}{\omega}X_1+X_5$$

admits a complete system of 12 invariants for which the conditions (1.3) are satisfied. The invariant solutions have the form:

$$u_1(t,x)=\mathcal{U}_1(x),\ \rho(t,x)=\mathcal{R}(x),\ p^m(t,x)=\mathcal{P}(x),$$

$$E^r(t,x)=\mathcal{E}(x),\ S_1^r(t,x)=\mathcal{S}_1(x),$$

$$u_2(t,x)=\mathcal{U}_2(x)\cos\omega t+\mathcal{U}_3(x)\sin\omega t,$$

$$u_3(t,x)=-\mathcal{U}_2(x)\sin\omega t+\mathcal{U}_3(x)\cos\omega t,$$

$$B_2(t,x)=\mathcal{B}_2(x)\cos\omega t+\mathcal{B}_3(x)\sin\omega t,$$

$$B_3(t,x)=-\mathcal{B}_2(x)\sin\omega t+\mathcal{B}_3(x)\cos\omega t,$$

$$S_2(t,x)=\mathcal{S}_2(x)\cos\omega t+\mathcal{S}_3(x)\sin\omega t,$$

$$S_3(t,x)=-\mathcal{S}_2(x)\sin\omega t+\mathcal{S}_3(x)\cos\omega t,$$

where the functions: $\mathcal{U}_1(x)$, $\mathcal{U}_2(x)$, $\mathcal{U}_3(x)$, $\mathcal{R}(x)$, $\mathcal{P}(x)$, $\mathcal{B}_2(x)$, $\mathcal{B}_3(x)$, $\mathcal{S}_2(x)$, and $\mathcal{S}_3(x)$ satisfy the system:

$$\mathcal{R}\mathcal{U}_1\mathcal{U}_1'+\mathcal{P}'+\frac{1}{\mu}\left(\mathcal{B}_2\mathcal{B}_2'+\mathcal{B}_3\mathcal{B}_3'\right)=\frac{\alpha}{c^2}\mathcal{S}_1,\ \mathcal{R}\mathcal{U}_1'+\mathcal{U}_1\mathcal{R}'=0,$$

$$\mathcal{R}\mathcal{U}_1\mathcal{U}_2'-\frac{B}{\mu}\mathcal{B}_2'=-\omega\mathcal{R}\mathcal{U}_3,\ \mathcal{R}\mathcal{U}_1\mathcal{U}_3'-\frac{B}{\mu}\mathcal{B}_3'=\omega\mathcal{R}\mathcal{U}_2,$$

$$\left[\gamma\mathcal{P}+\frac{4}{3}(\gamma-1)\mathcal{E}\right]\mathcal{U}_1'+\mathcal{U}_1\mathcal{P}_1+(\gamma-1)\mathcal{U}_1\mathcal{E}'=(\gamma-1)\alpha\left(\mathcal{E}-aT^4\right),$$

$$\mathcal{B}_2\mathcal{U}_1'-B\mathcal{U}_2'+\mathcal{U}_1\mathcal{B}_2'=-\omega\mathcal{B}_3,\ \mathcal{B}_3\mathcal{U}_1'-B\mathcal{U}_3'+\mathcal{U}_1\mathcal{B}_3'=\omega\mathcal{B}_2,$$

$$\mathcal{E}'=-\frac{3\alpha}{c^2}\mathcal{S}_1,\ \mathcal{S}_1'=-\alpha\left(\mathcal{E}-aT^4\right),\ \omega\mathcal{S}_2-\alpha\mathcal{S}_3=0,\ \alpha\mathcal{S}_2+\omega\mathcal{S}_3=0.$$

Because $\omega^2 + \alpha^2 \neq 0$, the last two equations give: $\mathcal{S}_2 = \mathcal{S}_3 = 0$.

Let $\mathcal{U}$ be the vector with the coordinates $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{E}$, $\mathcal{S}_1$. Then the previous system can be written as:

$$A(\mathcal{U})\,\mathcal{U}' = B(\mathcal{U}). \tag{3.1}$$

Let $a_r$ be the local radiative sound velocity given by

$$a_r^2 = \frac{1}{\rho}\left(\gamma p^m + \frac{4}{3}(\gamma - 1)E^r\right),$$

and

$$b_i = \frac{1}{\sqrt{\mu\rho}}B_i,$$

the Alfvén velocity. If

$$\left(\mathcal{U}_1^2 - b_1^2\right)\left[\mathcal{U}_1^4 - \left(a_r^2 + b^2\right)\mathcal{U}_1^2 + b_1^2 a_r^2\right] \neq 0,$$

then the matrix $A(\mathcal{U})$ has an inverse, and the system (3.1) takes the normal form:

$$\mathcal{U}' = A^{-1}(\mathcal{U})\,B(\mathcal{U}).$$

$2^0$. The subgroup $H$ admitted by the Lie algebra determined by the infinitesimal operator

$$X = \frac{1}{\omega}X_1 + \frac{1}{k}X_2 + X_5$$

admits a complete system of 12 invariants for which the conditions (1.3) are satisfied. Denote $\tau = \omega t - kx$. The invariant solutions have the form:

$$u_1(t,x) = \mathcal{U}_1(\tau),\ \rho(t,x) = \mathcal{R}(\tau),\ p^m(t,x) = \mathcal{P}(\tau),$$
$$E^r(t,x) = \mathcal{E}(\tau),\ S_1^r(t,x) = \mathcal{S}_1(\tau),$$
$$u_2(t,x) = \mathcal{U}_2(\tau)\cos\omega t + \mathcal{U}_3(\tau)\sin\omega t,$$
$$u_3(t,x) = -\mathcal{U}_2(\tau)\sin\omega t + \mathcal{U}_3(\tau)\cos\omega t,$$
$$B_2(t,x) = \mathcal{B}_2(\tau)\cos\omega t + \mathcal{B}_3(\tau)\sin\omega t,$$
$$B_3(t,x) = -\mathcal{B}_2(\tau)\sin\omega t + \mathcal{B}_3(\tau)\cos\omega t,$$
$$S_2(t,x) = \mathcal{S}_2(\tau)\cos\omega t + \mathcal{S}_3(\tau)\sin\omega t,$$
$$S_3(t,x) = -\mathcal{S}_2(\tau)\sin\omega t + \mathcal{S}_3(\tau)\cos\omega t,$$

where the functions: $\mathcal{U}_1(\tau)$, $\mathcal{U}_2(\tau)$, $\mathcal{U}_3(\tau)$, $\mathcal{R}(\tau)$, $\mathcal{P}(\tau)$, $\mathcal{B}_2(\tau)$, $\mathcal{B}_3(\tau)$, $\mathcal{S}_2(\tau)$, and $\mathcal{S}_3(\tau)$ satisfy the system:

$$(\omega - k\mathcal{U}_1)\,\mathcal{U}_1' - \frac{k}{\mathcal{R}}\mathcal{P}' - \frac{k}{\mu\mathcal{R}}\left(\mathcal{B}_2\mathcal{B}_2' + \mathcal{B}_3\mathcal{B}_3'\right) = \frac{\alpha}{c^2\mathcal{R}}\mathcal{S}_1,\ -k\mathcal{R}\mathcal{U}_1' + (\omega - k\mathcal{U}_1)\,\mathcal{R}' = 0,$$

$$(\omega - k\mathcal{U}_1)\,\mathcal{U}_2' + \frac{kB}{\mu\mathcal{R}}\mathcal{B}_2' = -\omega\mathcal{U}_3,\ (\omega - k\mathcal{U}_1)\,\mathcal{U}_3' - \frac{kB}{\mu\mathcal{R}}\mathcal{B}_3' = \omega\mathcal{U}_2,$$

$$-ka_r^2\mathcal{U}_1' + (\omega - k\mathcal{U}_1)\mathcal{P}_1 - k(\gamma-1)\mathcal{U}_1\mathcal{E}' = (\gamma-1)\alpha\left(\mathcal{E} - aT^4\right),$$

$$-k\mathcal{B}_2\mathcal{U}_1' + Bk\mathcal{U}_2' + (\omega - k\mathcal{U}_1)\mathcal{B}_2' = -\omega\mathcal{B}_3,\ -k\mathcal{B}_3\mathcal{U}_1' + Bk\mathcal{U}_3' + (\omega - k\mathcal{U}_1)\mathcal{B}_3' = \omega\mathcal{B}_2,$$

$$\omega\mathcal{E}' - k\mathcal{S}_1' = -\alpha\left(\mathcal{E} - aT^4\right),\ \omega\mathcal{S}_1' - k\frac{c^2}{3}\mathcal{E}' = -\alpha\mathcal{S}_1,$$

$$\omega\mathcal{S}_2' = -\alpha\mathcal{S}_2 - \omega\mathcal{S}_3,\ \omega\mathcal{S}_3' = \omega\mathcal{S}_2 - \alpha\mathcal{S}_3.$$

Let $\mathcal{U}$ be the vector with the coordinates $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{E}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$. Then the previous system can be written as:

$$A(\mathcal{U})\mathcal{U}' = B(\mathcal{U}). \tag{3.2}$$

If

$$\left(\frac{\omega^2}{k^2} - \frac{c^2}{3}\right)\left[\left(\mathcal{U}_1 - \frac{\omega}{k}\right)^2 - b_1^2\right]\left[\left(\mathcal{U}_1 - \frac{\omega}{k}\right)^4 - \left(a_r^2 + b^2\right)\left(\mathcal{U}_1 - \frac{\omega}{k}\right)^2 + a_r^2 b_1^2\right] \neq 0,$$

then the system (3.2) takes the normal form:

$$\mathcal{U}' = A^{-1}(\mathcal{U})\,B(\mathcal{U}).$$

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