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CONTROLLABILITY ON INIFINITE TIME HORIZON OF SECOND ORDER DELAY INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we shall establish sufficient conditions for the controllability on infinite time horizon of second order delay integrodifferential inclusions in Banach spaces with nonlocal conditions. We shall rely of a fixed point theorem due to Ma, which is an extension to multivalued maps, on locally convex topological spaces, of Schaefer's theorem.

Key words: Controllability, Convex multivalued map, Mild solution, Nonlocal Condition, Fixed point.
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1 Introduction

In this paper, we shall establish sufficient conditions on infinite time horizon for the controllability of second order delay integrodifferential inclusions in Banach spaces with nonlocal initial conditions. More precisely we consider the following semilinear system of the form

$$y'' - Ay \in \int_0^t K(t,s)F(s, y(\sigma(s)))ds + (Bu)(t), \quad t \in J := [0, \infty), \tag{1.1}$$

$$y(0) + f(y) = y_0, \; y'(0) = y_1 \tag{1.2}$$

where $F : J \times E \longrightarrow 2^E$ is a bounded, closed, convex valued multivalued map, $\sigma : J \to J$ is a continuous function such that $\sigma(t) \leq t, \forall t \in J$, $K : D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, $f \in C(J, E)$, A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a Banach space $E = (E, |\cdot|)$, $y_0, y_1 \in E$. Also the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. Finally B is a bounded linear operator from U to E.

The pioneering work on nonlocal evolution Cauchy problems is due to Byszewski. As pointed out by Byszewski [6], [5] the study of IVP with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. In fact, more authors have paid attention to the research of initial value problems with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [3], Byszewski [5], [6], Ntouyas [20] and Ntouyas and Tsamatos [18], [19].

Initial value problems for second order semilinear equations with nonlocal conditions, was studied in Ntouyas and Tsamatos [19] and Ntouyas [20].

On the other hand, controllability results of nonlinear integrodifferential systems in Banach spaces, by using the Schauder fixed point theorem, was studied by Balachandran, Balasubramaniam and Dauer in [1]. Han and Park [12], by using a Banach fixed point theorem, proved boundary controllability of differential equations with nonlocal conditions. Controllability results on semilinear differential inclusions, on compact intervals in Banach spaces, was studied by the authors in [4] by using a fixed point theorem for condensing maps due to Martelli [17].

In this paper, we define a new notion, *the nonlocal infinite controllability*, and study the controllability of systems (1.1)–(1.2) relied on a fixed point theorem due to Ma [16], which is an extension on locally convex topological spaces, of Schaefer's theorem.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. J_m is the compact real interval $[0, m]$ ($m \in \mathbb{N}$).

$C(J, E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Corduneanu [8])

$$d(y, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y - z\|_m}{1 + \|y - z\|_m} \quad \text{for each} \quad y, z \in C(J, E),$$

where

$$\|y\|_m := \sup\{|y(t)| : t \in J_m\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E. A measurable function $y : J \longrightarrow E$ is Bochner integrable, if and only if, $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [23]).

$L^1(J, E)$ denotes the linear space of equivalence classes of measurable functions $y : J \longrightarrow E$ such that $\int_0^{\infty} |y(s)|\, ds < \infty$.

V_p denotes the neighbourhood of 0 in $C(J, E)$ defined by

$$V_p := \{y \in C(J, E) : \|y\|_m \leq p \text{ for each } m \in \mathbb{N}\}.$$

The convergence in $C(J, E)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J, E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C(J_m, E)$ as $j \longrightarrow \infty$.

$M \subseteq C(J, E)$ is a bounded set if and only if there exists a positive function $\varphi \in C(J, \mathbb{R})$ such that

$$|y(t)| \leq \varphi(t) \text{ for all } t \in J \text{ and all } y \in M.$$

A set $M \subseteq C(J, E)$ is compact if and only if for each $m \in \mathbb{N}$, M is a compact set in the Banach space $(C(J_m, E), \|\cdot\|_m)$.

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(D) = \cup_{x\in D} G(x)$ is bounded in X for any bounded set D of X (i.e. $\sup\limits_{x\in D}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set V of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq V$.

G is said to be completely continuous if $G(D)$ is relatively compact for every bounded subset $D \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(X)$ is said to be measurable if for each $x \in X$ the distance between x and $G(t)$ is a measurable function on J. For more details on multivalued maps see the books of Deimling [9] and Hu and Papageorgiou [14].

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if
(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [11], Heikkila and Lakshmikantham [13], Fattorini [10], and to the papers of Travis and Webb [21], [22].

Definition 2.1. *A function $y \in C(J, E)$ is said to be a mild solution of (1.1)-(1.2) on J if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(\sigma(t)))$ a.e. on J, $y(0) + f(y) = y_0$, and*

$$y(t) = C(t)y_0 - C(t)f(y) + S(t)y_1 + \int_0^t S(t-s)(Bu)(s)\,ds$$

$$+ \int_0^t S(t-s)\int_0^s K(s,\tau)v(\tau)d\tau ds.$$

Definition 2.2. *The system (1.1)–(1.2) is said to be nonlocally infinite controllable on the interval* J, *if for every* $y_0, y_1, x_1 \in E$ *and for each* $m > 0$ *there exists a control* $u \in L^2(J_m, U)$, *such that the mild solution* $y(t)$ *of (1.1)–(1.2) satisfies* $y(m) + f(y) = x_1$.

We will need the following assumptions:

(H1) A is the infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$ and there exists $M > 0$ such that $M = \sup\{|C(t)|; t \in J\}$.

(H2) $F : J \times E \longrightarrow BCC(E); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \Big\{ g \in L^1(J, E) : \ g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } \ t \in J \Big\}$$

is nonempty.

(H3) the function f is completely continuous and there exists a constant L such that

$$|f(y)| \leq L, \quad \text{for each} \quad y \in E;$$

(H4) for each $t \in J_m$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J_m;

(H5) the map $t \longmapsto K_t$ is continuous from J_m to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t, s)$.

(H6) $\sigma : J \to J$ is a continuous function, such that $\sigma(t) \leq t, \forall t \in J$.

(H7) for each $m > 0$ the linear operator $W : L^2(J_m, U) \to E$, defined by

$$Wu = \int_0^m S(m - s)Bu(s)\, ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J_m, U) \backslash ker W$ and there exist positive constants M_1 and M_2 such that $|B| \leq M_1$ and $|W^{-1}| \leq M_2$.

(H8) $\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_c^\infty \frac{du}{\psi(u)} = \infty;$$

where $c = M(|y_0| + L + Mm|y_1| + M_0)$, and

$$M_0 = mM_1M_2\Big[|x_1| + L + M|y_0| + ML + mM|y_1| \\ + mM \sup_{t \in J_m} K(t) \int_0^m p(s)\psi(|y(s)|)ds\Big].$$

(H9) for each neighbourhood V_p of 0, $y \in V_p$ and $t \in J_m$ the set

$$\Big\{C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g(u)du\,ds : g \in S_{F,y}\Big\}$$

is relatively compact.

Remark. (i) If $\dim E < \infty$ and J is compact, then for each $y \in C(J,E)$ $S_{F,y} \neq \emptyset$ (see Lasota and Opial [15]).

(ii) If $\dim E = \infty$ and J is compact then $S_{F,y}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t,y)\}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [14]).

(iii) Examples with $W : L^2(J_m,U) \to E$ such that W^{-1} exists and is bounded are discussed in [7].

(iv) If we assume that $C(t),\ t \in J$ is completely continuous then (H9) is satisfied.

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.1. *[15] Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow BCC(C(I,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2. *[16]. Let X be a locally convex space and $N : X \longrightarrow 2^X$ be a compact convex valued, u.s.c. multivalued map such that for every closed neighbourhood V_p of 0, $N(V_p)$ is a relatively compact set for each $p \in \mathbb{N}$. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \ \text{ for some } \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3 Main Result

Theorem 3.1. *Let $f : C(J,E) \longrightarrow E$ be a continuous function. Assume that hypotheses (H1)–(H9) are satisfied. Then the problem (1.1)–(1.2) is nonlocally infinite controllable on J.*

PROOF. Using hypothesis (H7) for an arbitrary function $y(\cdot)$ define the control

$$u_y^m(t) = W^{-1}\Big[x_1 - f(y) - C(m)y_0 + C(m)f(y) - S(m)y_1$$

$$-\int_0^m S(m-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds\Big](t)$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1(J,E) : g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J \Big\}.$$

We shall now show that when using this control, the operator $N : C(J,E) \longrightarrow 2^{C(J,E)}$ defined by

$$N(y) := \Big\{ h \in C(J,E) : h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)(Bu_y^m)(s)\,ds$$

$$+ \int_0^t S(t-s) \int_0^s K(s,\tau) g(\tau)\, d\tau ds : g \in S_{F,y} \Big\}$$

has a fixed point. This fixed point is then a solution of the system (1.1)–(1.2).

Clearly $x_1 - f(y) \in (Ny)(m)$.

We shall show that $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C(J,E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

This step is obvious. However, for completness, we give the proof. If $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J_m$ we have

$$h_i(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)(Bu_y^m)(s)\,ds$$

$$+ \int_0^t S(t-s) \int_0^s K(s,\tau) g_i(\tau) d\tau ds,\ i = 1, 2.$$

Let $0 \le \alpha \le 1$. Then for each $t \in J_m$ we have

$$(\alpha h_1 + (1-\alpha) h_2)(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)(Bu_y^m)(s)\,ds$$

$$+ \int_0^t S(t-s) \int_0^s K(s,\tau)[\alpha g_1(\tau) + (1-\alpha) g_2(\tau)] d\tau ds.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha) h_2 \in N(y).$$

Step 2: *$N(V_q)$ is bounded in $C(J,E)$ for each $q \in \mathbb{N}$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N(y), y \in V_q$ one has $||h||_m \le \ell_m$.

If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)(Bu_y^m)(s)\,ds$$

$$+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t \in J.$$

By (H1), (H3)–(H8) we have for each $t \in J_m$ that

$$\begin{aligned}
|h(t)| &\leq |C(t)||y_0| + |C(t)||f(y)| + |S(t)||y_1| + \left\| \int_0^t S(t-s)(Bu_y^m)(s)\,ds \right\| \\
&\quad + \left\| \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds \right\| \\
&\leq M|y_0| + ML + Mm|y_1| \\
&\quad + mMM_1M_2[|x_1| + L + M|y_0| + ML + mM|y_1| \\
&\quad + Mm \sup_{t\in J_m} K(t)||p||_{L^1} \sup_{t\in J_m} \psi(|y(t)|)] \\
&\quad + M\int_0^t \int_0^s |K(s,u)|p(u)\psi(|y(\sigma(u))|)duds \\
&\leq M|y_0| + ML + mM|y_1| \\
&\quad + mMM_1M_2[|x_1| + L + M|y_0| + ML + mM|y_1| \\
&\quad + Mm \sup_{t\in J_m} K(t)||p||_{L^1} \sup_{t\in J_m} \psi(|y(t)|)] \\
&\quad + Mm \sup_{t\in J_m} K(t)|p|_{L^1} \sup_{t\in J_m} \psi(|y(t)|) = l_m.
\end{aligned}$$

Then for each $h \in N(V_q)$ we have $||h||_m \leq l_m$.

Step 3: *For each $q \in \mathbb{N}$, $N(V_q)$ is equicontinuous for $V_q \in C(J,E)$.*

Let $t_1, t_2 \in J_m$, $t_1 < t_2$ and V_q be a neighbourhood of 0 in $C(J,E)$ for $q \in \mathbb{N}$. For each $y \in V_q$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)(Bu_y^m)(s)ds$$

$$+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t \in J.$$

Thus

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq |(C(t_2)-C(t_1))y_0| + L|C(t_2)-C(t_1)| \\
&\quad +|S(t_2y_1 - S(t_1)y_1| \\
&\quad \Big\|\int_0^{t_2}\Big[S(t_2-s)-S(t_1-s)\Big]BW^{-1}\Big[x_1 - f(y) - C(m)y_0 \\
&\quad +C(m)f(y) - S(m)y_1 - \int_0^m S(m-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big](\eta)d\eta\Big\| \\
&\quad +\Big\|\int_{t_1}^{t_2} S(t_1-s)BW^{-1}\Big[x_1 - f(y) - C(m)y_0 + C(m)f(y) - S(m)y_1 \\
&\quad -\int_0^m S(m-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big](\eta)d\eta\Big\| \\
&\quad +\Big\|\int_0^{t_2}\Big[S(t_2-s)-S(t_1-s)\Big]\int_0^s K(s,\tau)g(\tau)d\tau ds\Big\| \\
&\quad +\Big\|\int_{t_1}^{t_2} S(t_1-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big\| \\
&\leq |(C(t_2)-C(t_1))y_0| + L|C(t_2)-C(t_1)| \\
&\quad +|S(t_2y_1 - S(t_1)y_1| \\
&\quad +\int_0^{t_2}\Big|S(t_2-s)-S(t_1-s)\Big|M_1M_2\Big[|x_1| + L + M|y_0| \\
&\quad +ML + mM|y_1| + Mm\sup_{t\in J_m} K(t)\int_0^b p(s)\psi(|y(s)|)ds\Big](\eta)d\eta \\
&\quad +\int_{t_1}^{t_2}|S(t_1-s)|M_1M_2\Big[|x_1| + L + M|y_0| + ML + mM|y_1| \\
&\quad +Mm\sup_{t\in J_m} K(t)\int_0^m p(s)\psi(|y(s)|)ds\Big](\eta)d\eta \\
&\quad +\sup_{t\in J_m} K(t)\Big\|\int_0^{t_2}\Big[S(t_2-s)-S(t_1-s)\Big]\int_0^s g(\tau)d\tau ds\Big\| \\
&\quad +M\sup_{t\in J_m} K(t)(t_2-t_1)\int_0^m \|g(s)\|ds.
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3, (H9) and the definition of the metric of the Fréchet space $C(J,E)$, we can conclude that N is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = C(t)y_0 - C(t)f(y_n) + S(t)y_1 + \int_0^t S(t-s)(Bu^m_{y_n})(s)ds$$
$$+ \int_0^t S(t-s)\int_0^s K(s,\tau)g_n(\tau)d\tau ds, \quad t \in J,$$

where

$$u^m_{y_n}(t) = W^{-1}\Big[x_1 - f(y_n) - C(m)y_0 + C(m)f(y_n)$$
$$- \int_0^m S(m-s)\int_0^s K(s,\tau)g_n(\tau)d\tau ds\Big](t).$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) &= C(t)y_0 - C(t)f(y_*) + S(t)y_1 + \int_0^t S(t-s)(Bu^m_{y_*})(s)ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds, \quad t \in J, \end{aligned} \tag{3.1}$$

where

$$u^m_{y_*}(t) = W^{-1}\Big[x_1 - f(y_*) - C(m)y_0 + C(m)f(y_*)$$
$$- \int_0^m S(m-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds\Big](t).$$

Set

$$\overline{u}^m_y(t) = W^{-1}\Big[x_1 - f(y) - C(m)y_0 + C(m)f(y)\Big](t).$$

The idea is then to use the facts that

(i) $h_n \longrightarrow h_*$;

(ii) $h_n - C(t)y_0 + C(t)f(y_n) - S(t)y_1 - \int_0^t S(t-s)(B\overline{u}^m_{y_n})(s)ds \in \Gamma(S_{F,y_n})$ where

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

defined by

$$g \longmapsto \Gamma(g)(t) = \int_0^t S(t-s)\int_0^s K(s,\tau)\Big[BW^{-1}\Big(\int_0^m S(m-\sigma)g(\sigma)d\sigma\Big)(\tau) + g(\tau)\Big]d\tau ds.$$

If $\Gamma \circ S_F$ is a closed graph operator, we would be done. But we do not know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cut the functions $y_n, h_n - C(t)y_0 + C(t)f(y_n) - S(t)y_1 - \int_0^t S(t-s)(B\overline{u}^m_{y_n})(s)ds, g_n$ and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbb{N} \cup \{0\}$. Then, using Lemma 2.2, in this case we are able to affirm that (3.1) is true on the compact interval $[k, k+1]$, i.e.

$$\begin{aligned} h_*(t)\Big|_{[k,k+1]} &= C(t)y_0 - C(t)f(y_*) + S(t)y_1 + \int_0^t S(t-s)(Bu_{y_*^m})(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_*^k(\tau)d\tau ds \end{aligned}$$

for a suitable L^1-selection g_*^k of $F(t, y_*(t))$ on the interval $[k, k+1]$.

At this point we can paste the functions g_*^k obtaining the selection g_* defined by

$$g_*(t) = g_*^k(t) \quad \text{for} \quad t \in [k, k+1).$$

We obtain then that g_* is a L^1-selection and (3.1) will satisfied.

We give now the details.

Since f, W^{-1} are continuous, then $\overline{u}^m_{y_n}(t) \longrightarrow \overline{u}^m_{y_*}(t)$ for $t \in J$.

Clearly we have that

$$\begin{aligned} &\Big|\Big(h_n - C(t)y_0 + C(t)f(y_n) - S(t)y_1 - \int_0^t S(t-s)(B\overline{u}^m_{y_n})(s)ds \\ &\qquad - \Big(h_* - C(t)y_0 + C(t)f(y_*) - S(t)y_1 - \int_0^t S(t-s)(B\overline{u}^m_{y_*})(s)ds\Big)\Big|_\infty \to 0, \end{aligned}$$

as $n \to \infty$.

Now, we consider for all $k \in \mathbb{N} \cup \{0\}$, the mapping

$$S_F^k : C([k, k+1], E) \longrightarrow L^1([k, k+1], E)$$

$$u \longmapsto S^k_{F,u} := \{h \in L^1([k, k+1], E) : h(t) \in F(t, u(t)) \quad \text{for a.e.} \quad t \in [k, k+1]\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k, k+1], E) \longrightarrow C([k, k+1], E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t S(t-s)\int_0^s K(s,\tau)\Big[BW^{-1}\Big(\int_0^m S(m-\sigma)g(\sigma)d\sigma\Big)(\tau) + g(\tau)\Big]d\tau ds.$$

Clearly, Γ is linear and continuous. Indeed one has

$$|\Gamma g|_\infty \le m^2 M \sup_{t \in [k,k+1]} |K(t)|(m^2 M M_1 M_2 + 1)\|g\|_{L^1}.$$

From Lemma 2.2, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - C(t)y_0 + C(t)f(y_n) - S(t)y_1 - \int_0^t S(t-s)(B\overline{u}^m_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.2 that

$$\begin{aligned} h_*(t) - C(t)y_0 + C(t)f(y_*) - S(t)y_1 &- \int_0^t S(t-s)(B\overline{u}^m_{y_*})(s)ds \\ &= \int_0^t S(t-s)\int_0^s \Big[W^{-1}\Big(\int_0^m S(m-\sigma)g_*(\sigma)d\sigma\Big)(\tau) + g_*(\tau)\Big]d\tau ds \end{aligned}$$

for some $g_* \in S_{F,y_*}$.

Step 5: *The set*

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y), \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)y_0 - \lambda^{-1}C(t)f(y) + \lambda^{-1}S(t)y_1 \\ &+ \lambda^{-1}\int_0^t S(t-s)BW^{-1}\Big[x_1 - f(y) - C(m)y_0 + C(m)f(y) - S(t)y_1 \\ &- \int_0^m S(m-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](\eta)d\eta \\ &+ \lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \ t \in J. \end{aligned}$$

This implies by (H1), (H3)–(H8) that for each $t \in J_m$ we have

$$\begin{aligned} |y(t)| &\le M|y_0| + ML + mM|y_1| \\ &\quad + mMM_1M_2\Big[|x_1| + L + M|y_0| + ML + mM|y_1| \\ &\quad + mM\sup_{t\in J_m} K(t)\int_0^m p(s)\psi(|y(\sigma(s))|)ds\Big] \\ &\quad + M\left\|\int_0^t\int_0^s K(s,\tau)g(\tau)d\tau ds\right\| \\ &\le M|y_0| + ML + mM|y_1| \end{aligned}$$

$$+mMM_1M_2\Big[|x_1| + L + M|y_0| + ML + mM|y_1|$$

$$+mM \sup_{t\in J_m} K(t)\int_0^m p(s)\psi(|y(\sigma(s))|)ds\Big]$$

$$+Mm \sup_{t\in J_m} K(t)\int_0^t p(s)\psi(|y(\sigma(s))|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M|y_0| + ML + MM_0,\ \ |y(t)| \leq v(t),\ \ t \in J_m$$

and

$$v'(t) = Mm \sup_{t\in J_m} K(t)p(t)\psi(|y(\sigma(t))|),\ \ t \in J_m.$$

Using the nondecreasing character of ψ and the fact that $\sigma(t) \leq t, \forall t \in J_m$ we get

$$v'(t) \leq Mm \sup_{t\in J_m} K(t)p(t)\psi(v(t)),\ \ t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq Mm \sup_{t\in J_m} K(t)\int_0^m p(s)ds < \infty.$$

From (H8) it follows that there exists a constant d such that $v(t) \leq d,\ t \in J_m$, and hence $\|y\|_m \leq d$ where d depends only on m and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 2.2 we deduce that N has a fixed point and thus the system (1)–(2) is nonlocally infinite controllable on J. □

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