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CONTROLLABILITY OF SEMILINEAR EVOLUTION EQUATIONS WITH TIME LAGS

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Abstract. Sufficient conditions for controllability of semilinear evolution equations with time lags in Banach space are established. The results are obtained by using semigroups of linear operators, Hölder continuous functions and Schauder's fixed point theorem. An application to partial integrodifferential equations is given.

Keywords: Controllability, evolution equations, fixed point theorem.

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1 Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional space has been extensively studied. Several authors have extended the concept to infinite dimensional systems represented by the evolution equations with bounded linear operators in Banach spaces [8–10]. Recently Dauer and Balasubramaniam [1] established sufficient conditions for the null controllability of semilinear integrodifferential systems with unbounded operators and infinite delay in Banach space. Certain problems in control of fluid flow [2] can be modelled by the semilinear system

$$\frac{\partial x(t)}{\partial t} + Ax(t) + S(x(t)) = f(x(t), u(t)),\ t \in J = [0, T] \tag{1.1}$$

$$x(0) = x_0 \in X \tag{1.2}$$

in a Banach space X, where the operator A is a closed, densely defined operator. Let $0 \in \sigma(A)$, replace A by $A + CI$, and S by $S - CI$ for some $C > 0$. Under this condition, the fractional powers A^α are bounded for $\alpha < 0$. In this paper assume that (1.1)–(1.2) is the Navier-Stokes system in a domain in R^2 or R^3 with a control term, so that $Ax = -P\Delta x$ (boundary conditions included in the domain of Δ). $Sx = P(x \cdot \nabla x)$, P the projection operator of L^2 into the space of divergence-free vectors that are parallel to the boundary and die down at infinity in the unbounded case.

The actual flow control problems leading to this kind of model and the same modelled equation have been discussed in [3]. The Navier-Stokes system in two dimensions (see [7]) is a special case of our general result and will be discussed in the example. The purpose of this paper is to extend the use of fixed point theorems to semilinear evolution equations with time lags where the function $A(\cdot)$ is Hölder continuous. The considered system is an abstract formulation of the parabolic semilinear second order partial differential equations discussed in [4, 12].

2 Preliminaries

Consider the semilinear delay differential system in the Banach space of the form

$$\begin{aligned} \dot{x}(t) + A(t)x(t) &= f(t, x_t, x(t)) + (Bu)(t), \ t \in J = [0, T] \\ x(t) &= \phi(t), \ t \in [-r, 0] \end{aligned}$$

where $x(\cdot)$ is the state in the Banach space X and the control function $u(\cdot)$ is given in L^2(J,V), a Banach space of admissible control functions with V as a Banach space. B is a bounded linear operator mapping from V into X, $f : R \times X_\alpha \times X \to X$ is a continuous nonlinear operator and $\{A(t) : t \in J\}$ is a family of closed, densely defined linear operators in X satisfying the following assumptions.

(a) The domain $D(A(t))$ of $A(t)$ is independent of t.

(b) For each $t \in J$, the resolvent $R(\lambda, A(t))$ exists for all $\lambda \geq 0$ and

$$\|R(\lambda, A(t))\| \leq k(1 + |\lambda|)^{-1}$$

where k is some constant independent of λ and t.

(c) The function $A(\cdot) : J \to L_b(X_1, X)$ is Hölder continuous. Here $L_b(X_1, X)$ denotes the space of bounded linear operators from X_1 to X furnished with the uniform operator topology.

This implies that the operator $A(t)$ has an inverse $A^{-1}(t) \in L_b(X, X)$. Further, $A(0) = A$ and $\|x\|_1 = \|Ax\|$ for $x \in D(A)$. Consequently $X_1 = (D(A), \|\cdot\|_1)$ is a Banach space and $X_1 \hookrightarrow X$. The fractional power operator $A^\alpha(t), \alpha \in (0,1)$, has a dense domain $D(A^\alpha(t))$, which is independent of t. Let $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in D(A^\alpha)$ and denote by X_α the Banach space $(D(A^\alpha), \|\cdot\|_\alpha)$. Then it is clear that $X_\beta \hookrightarrow X_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$ (see [6]).

Definition 2.1. *A function $f : J \to X$ is said to be α-Hölder continuous if there exists a constant $L \geq 0$ such that*

$$\|f(t) - f(s)\| \leq L|t - s|^\alpha \text{ for all } s, t \in J = [0, T].$$

We denote by $C^{0,\alpha}(J; X)$ the space of α-Hölder continuous functions from J into X.

For the initial value problem

$$\dot{x}(t) + A(t)x(t) = f(t) + (Bu)(t), \ t \in J \tag{2.1}$$

$$x(0) = x_0 \tag{2.2}$$

the solution $x(\cdot)$ exists for every Hölder continuous right-hand side f (see [12]). Moreover, $x \in C^1(J, X)$ provided $x \in D(A)$. Further, there exists a unique evolution operator $U(t, \tau) \in L(X, X)$, $0 \leq \tau \leq t \leq T$ such that every solution of (2.1)–(2.2) can be represented in the form

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, \tau)f(\tau)d\tau + \int_0^t U(t, \tau)(Bu)(\tau)d\tau, \ t \in J.$$

Let $C_{\delta,\alpha} = C([-\gamma, -\delta], X_\alpha)$ denote the Banach space of continuous X_α-valued functions on $[-\gamma, -\delta]$ with supremum norm where $0 \leq \delta \leq \gamma$, $\alpha \in$(0,1). If x is a continuous function from $[-\gamma, T]$ to X_α then, for each $t \in J$, x_t denotes an element of $C_{\delta,\alpha}$ given by $x_t(s) = x(t+s)$, $-\gamma \le s \le -\delta$.

Consider the abstract delay differential equation

$$\dot{x}(t) + A(t)x(t) = f(t, x_t, x(t)) + (Bu)(t), \ t \in J \tag{2.3}$$

$$x(t) = \phi(t), \ t \in [-r, 0] \tag{2.4}$$

Suppose that $0 \le \alpha \le \beta \le 1$. For the existence of solution of (2.3)–(2.4) there exists a constant K dependent on α, β, γ, such that the following conditions hold (see [8]).

1) $\|U(t,\tau)\|_{\alpha,\beta} \le K(\alpha, \beta)$

2) $\|U(t,\tau) - U(s,\tau)\|_{\alpha,\beta} \le K(\alpha,\beta,\gamma)|t-s|^\gamma, \ 0 \le \gamma < \beta - \alpha$

3) Define

$$\wp(\phi, f, B) = U(t,0)\phi(0) + \int_0^t U(t,\tau)f(\tau)d\tau + \int_0^t U(t,\tau)(Bu)(\tau)d\tau,$$

then $\wp$ is a continuous linear operator from $X_\beta \times C(J,X) \times C(J,V)$ into $C^\gamma(J, X_\alpha)$ for every $\gamma \in [0, \beta - \alpha)$.

4) There exists constants $\gamma_1, \gamma_2 \in (0,1]$ and for every $\rho \ge 0$ there is a constant $K(\rho) > 0$ such that

$$\|f(t,y,z) - f(s,y^*,z^*)\| \le K(\rho)(|t-s|^{\gamma_1} + |y-y^*|^{\gamma_2}_{C_{\delta,\alpha}} + \|z - z^*\|_\alpha)$$

for (t,y,z), $(s, y^*, z^*) \in I \times C_{\delta,\alpha} \times X\alpha$ satisfying

$$\|y\|_{C_{\delta,\alpha}} \le \rho, \ \ \|y*\|_{C_{\delta,\alpha}} \le \rho, \ \ \|z\|_\alpha \le \rho, \ \ \|z^*\|_\alpha \le \rho.$$

5) There exists a Banach space E with $X_\alpha \hookrightarrow E \hookrightarrow X$ and a constant $\sigma \in (1, 1/\alpha)$ such that

$$\|f(t,y,z)\| \le K(\rho)(1 + \|z\|^\sigma_\alpha)$$

for every $\rho \ge 0$ and (t, y, z) $\in J \times C_{\delta,\alpha} \times X_\alpha$ satisfying $\|y\|_{C_{\delta,\alpha}} \le r$ and $\|z\|_E \le \rho$.

6) A(t) has compact resolvent for all t $\in$ J.

The operator f is well-defined for each $t \in$ J. Further, for $x_t \in C_{\delta,\alpha}$ some $\delta > 0$ and $x_t \in C_{0,\alpha}$, the equation (2.3)–(2.4) is said to have positive arm of time lag and zero arm of time lag, respectively.

Suppose that the conditions (1)–(6) are satisfied, then there exists a unique solution of the equation (2.3)–(2.4) for every Hölder continuous right hand side f such that

$$x(t) = U(t,0)\phi(0) + \int_0^t U(t,\tau)f(\tau, x_\tau, x(\tau))d\tau + \int_0^t U(t,s)(Bu)(s)ds \text{ for } t \in J$$
$$x(t) = \phi(t), \ t \in [-r, 0]$$

Definition 2.2. *The system (2.3)–(2.4) is said to be null controllable on the interval J if for every continuous function $\phi \in C_\alpha$ there exists a control $u \in L^2(J,V)$ such that the solution $x(\cdot)$ of (2.3)–(2.4) satisfies $x(T) = 0$.*

3 Main result

Theorem 3.1. *Suppose that conditions (1)–(6) hold and the linear operator W from $L^2(J,V)$ into X defined by*

$$Wu = \int_0^T U(T,s)(Bu)(s)ds \tag{3.1}$$

has an invertible operator W^{-1} defined on $X\backslash\ker(W)$. If there exist positive constants N_1, N_2 such that $\|B\| \leq N_1, \|W^{-1}\| \leq N_2$, then the system (2.3)–(2.4) is null controllable on J.

PROOF. Using the hypothesis, define the control

$$u(t) = -W^{-1}[U(T,0)\phi(0) + \int_0^T U(T,\tau)f(\tau,x_\tau,x(\tau))d\tau](t).$$

Now it is shown that when using this control the operator defined by

$$\begin{aligned}(\Phi x)(t) &= \phi(t) \text{ for } t \in [-r,0] && (3.2)\\ (\Phi x)(t) &= U(t,0)\phi(0) - \int_0^t U(t,\mu)BW^{-1}\Big\{U(T,0)\phi(0)\\ &\quad + \int_0^T U(T,\tau)f(\tau,x_\tau,x(\tau))d\tau\Big\}(\mu)d\mu\\ &\quad + \int_0^t U(t,\tau)f(\tau,x_\tau,x(\tau))d\tau && (3.3)\end{aligned}$$

has a fixed point. This fixed point is a solution of equation (2.3)–(2.4).

Clearly $(\Phi x)(T) = 0$, which means that the control u steers the semilinear evolution system from the initial function ϕ to 0 in time T provided the nonlinear operator Φ has a fixed point.

Let $Y = C([-r,T];X)$ and

$$Y_0 = \{x \in Y : \|x(t)\| = \|\phi(t)\| \text{ on } t \in (-\infty,0] \text{ and } \|x(t)\| \leq r \text{ for } t \in J\}$$

where

$$\begin{aligned}r = {} & K(\alpha,\beta)\|\phi(0)\|\{1 + K(\alpha,\beta)N_1N_2T\}\\ & + K(\alpha,\beta)K(\rho)(1 + \|x(\tau)\|_\alpha^\sigma)T\{1 + K(\alpha,\beta)N_1N_2T\}\end{aligned}$$

Then Y_0 is clearly a bounded, closed, convex subset of Y. Further $\|(\Phi x)(t)\| = \|\phi(0)\|$ on $(-\infty, 0]$ and

$$\begin{aligned}
\|(\Phi x)(t)\| &\le \|U(t,0)\phi(0)\| + \int_0^t \|U(t,\mu)BW^{-1}\|\Big\{\|U(T,0)\phi(0)\| \\
&\quad + \int_0^t \|U(T,\tau)f(\tau,x_\tau,x(\tau))\|d\tau\Big\}(\mu)d\mu \\
&\quad + \int_0^t \|U(t,\tau)f(\tau,x_\tau,x(\tau))\|d\tau \\
&\le K(\alpha,\beta)\|\phi(0)\| + K(\alpha,\beta)N_1N_2\Big\{K(\alpha,\beta)\|\phi(0)\|T \\
&\quad + K(\alpha,\beta)K(\rho)(1+\|x(\tau)\|_\alpha^\sigma)T^2\Big\} \\
&\quad + K(\alpha,\beta)K(\rho)(1+\|x(\tau)\|_\alpha^\sigma)T \\
&\le K(\alpha,\beta)\ \|\phi(0)\|\{1+K(\alpha,\beta)N_1N_2T\} \\
&\quad + K(\alpha,\beta)K(\rho)(1+\|x(\tau)\|_\alpha^\sigma)T\{1+K(\alpha,\beta)N_1N_2T\} \\
&\le r, \qquad \text{for } t \in J.
\end{aligned}$$

It follows that Φ is also continuous and maps Y_0 into itself. Moreover Φ maps Y_0 into a precompact subset of Y_0. To prove this we first show that for every fixed $t \in J$ the set

$$Y_0(t) = \{(\Phi x)(t) : x \in Y_0\}$$

is precompact in X. This is clear for $t \in (-\infty, 0]$, since $Y_0(t) = \{\phi(t)\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$, define

$$\begin{aligned}
(\Phi_\epsilon x)(t) = U(t,0)\phi(0) - \int_0^{t-\epsilon} U(t,\mu)BW^{-1}\Big\{U(T,0)\phi(0) \\
+ \int_0^T U(T,\tau)f(\tau,x_\tau,x(\tau))d\tau\Big\}(\mu)d\mu \\
+ \int_0^{t-\epsilon} U(t,\tau)f(\tau,x_\tau,x(\tau))d\tau.
\end{aligned}$$

Since for each t, $A(t)$ is the generator of an analytic semigroup, its resolvent is compact, and the set

$$Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Y_0\}$$

is precompact in X for every ϵ, $0 < \epsilon < t$. Furthermore, for $x \in Y_0$ we have

$$\begin{aligned}
\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| &\le \|-\Big\{\int_{t-\epsilon}^t U(t,\mu)BW^{-1}[U(T,0)\phi(0) \\
&\quad + \int_0^T U(T,\tau)f(\tau,x_\tau,x(\tau))d\tau](\mu)\Big\}\|d\mu \\
&\quad + \|\int_{t-\epsilon}^t U(t,\tau)f(\tau,x_\tau,x(\tau))d\tau\| \\
&\le \epsilon K^2(\alpha,\beta)N_1N_2\Big\{\|\phi(0)\| + K(\rho)(1+\|x(t)\|_\alpha^\sigma)T\Big\} \\
&\quad + \epsilon K(\alpha,\beta)C(\rho)(1+\|x(\tau)\|_\alpha^\sigma)
\end{aligned}$$

which implies that $Y_0(t)$ is totally bounded; that is, $Y_0(t)$ is precompact in X.

We want to show that

$$\Phi(Y_0) = \{\Phi x : x \in Y_0\}$$

is an equicontinuous family of functions. For that, let $t_2 > t_1 > 0$. Then we have

$$\begin{aligned}
\|(\Phi x)\,(t_1) - (\Phi x)(t_2)\| &\le \|U(t_1,0) - U(t_2,0)\|\|\phi(0)\| + \\
&+ \Big\| \int_0^{t_1} U(t_1,\mu) B W^{-1}\Big\{U(T,0)\phi(0) + \int_0^T U(T,\tau) f(\tau, x_\tau, x(\tau))d\tau\Big\}(\mu)d\mu \\
&- \int_0^{t_2} U(t_2,\mu) B W^{-1}\Big\{U(T,0)\phi(0) + \int_0^T U(T,\tau) f(\tau, x_\tau, x(\tau))d\tau\Big\}(\mu)d\mu \Big\| \\
&+ \Big\| \int_0^{t_1} U(t_1,\tau) f(\tau, x_\tau, x(\tau))d\tau - \int_0^{t_2} U(t_2,\tau) f(\tau, x_\tau, x(\tau))d\tau \Big\| \\
&\le \|U(t_1,0) - U(t_2,0)\|\|\phi(0)\| + \\
&+ \int_0^{t_1} \|U(t_1,\mu) - U(t_2,\mu)\|\|BW^{-1}\|\Big\{\Big\|U(T,0)\phi(0) + \int_0^T U(T,\tau) f(\tau, x_\tau, x(\tau))d\tau\Big\}(\mu)\Big\| d\mu \\
&+ \int_{t_1}^{t_2} \|U(t_2,\mu)\|\|BW^{-1}\|\Big\{\|U(T,0)\phi(0) + \int_0^T U(T,\tau) f(\tau, x_\tau, x(\tau))d\tau\Big\}(\mu)\| d\mu \\
&+ \int_0^{t_1} \|U(t_1,\tau) - U(t_2,\tau)\|\|f(\tau, x_\tau, x(\tau))\|d\tau + \int_{t_1}^{t_2} \|U(t_2,\tau)\|\|f(\tau, x_\tau, x(\tau))\|d\tau \\
&\le K(\alpha,\beta,\gamma)|t_1 - t_2|^\gamma \|\phi(0)\| + \\
&+ K(\alpha,\beta,\gamma)|t_1 - t_2|^\gamma N_1 N_2 \{K(\alpha,\beta)\|\phi(0)\| + K(\alpha,\beta)K(\rho)(1 + \|x(\tau)\|_\alpha^\sigma)T\} \\
&+ \int_{t_1}^{t_2} \|\|U(t_2,\mu)\| N_1 N_2 \Big\{K(\alpha,\beta)\|\phi(0)\| + \int_0^T U(T,\tau) f(\tau, x_\tau, x(\tau))d\tau\Big\}(\mu)\| d\mu \\
&+ K(\alpha,\beta,\gamma)|t_1 - t_2|^\gamma K(\rho)(1 + \|x(\tau)\|_\alpha^\sigma)T + \int_{t_1}^{t_2} \|U(t_2,\tau)\|\|f(\tau, x_\tau, x(\tau))\|d\tau. \qquad (3.4)
\end{aligned}$$

Thus the right-hand side of (3.4), tends to zero as $t_1 - t_2 \to 0$. So, $\Phi(Y_0)$ is an equicontinuous family of functions. Also, $\Phi(Y_0)$ is bounded in Y, and so by the Arzela-Ascoli Theorem $\Phi(Y_0)$ is precompact. Hence from the Schauder fixed point theorem, Φ has a fixed point in Y_0 (see Pazy [11]). Any fixed point of Φ is a mild solution of (2.3)–(2.4) on J satisfying $(\Phi x)(t) = x(t) \in X$.

Thus, the system (2.3)–(2.4) is null controllable on J. □

4 Example

Consider the semilinear parabolic problem

$$\frac{\partial u}{\partial t} + \mathtt{A}''(t,x,D)u = g(t,x,Au_t,u,\nabla u) + (Bv)(t), \ (t,x) \in I \times \Omega, \qquad (4.1)$$

$$\mathtt{B}''(x,D)u = 0, \ (t,x) \in I \times \Gamma, \qquad (4.2)$$

where Ω is an open domain in R^n with its boundary Γ being an (n-1) dimensional $C^{2+\mu}$ monifold for some $\mu \in (0,1)$ such that Ω lies on one side of Γ. The spatial and the boundary operators A" and B" are defined as

$$\begin{aligned} \text{A}''(t,x,D)\phi \equiv & - \sum_{1\le i,j\le n} a_{ij}(t,x)D_iD_j\phi \\ & + \sum_{1\le i\le n} b_i(t,x)D_i\phi + C(t,x)\phi,\ (t,x)\in I\times\Omega, \\ \text{B}''(x,D)\phi = & \ b_0(x)\phi + c_0(x)(\partial\phi/\partial\gamma),\ (t,x)\in I\times\Gamma, \end{aligned}$$

where D_i denotes the spatial derivative with respect to x_i and $(\partial/\partial\gamma)$ the directional derivative along the outwardly directed $C^{1+\mu}$ vector field γ locally normal to Γ. For the spatial operator A", assume that all the coefficients belong to $C^{\mu,\mu/2}I\times\Omega)$ and that it is strongly elliptic. For the boundary operator B" assume that either $c_0 = 0$ and $b_0 = 1$ giving the standard Dirichlet problem or $c_0 = 1$ and $b_0 \in C^{1+\mu}(\Gamma)$ $(b_0 > 0)$ giving the standard third boundary problem.

The nonlinear function

$$g = g(t,x,\xi,\zeta,\eta),\ (t,x,\xi,\zeta,\eta)\in I\times\Omega\times R\times R\times R^n$$

is assumed to satisfy the following conditions:

i) $g: I\times\Omega\times R^{n+2}\to R$ is continuous.

ii) $g(\cdot,\cdot,\cdot,\zeta,\eta)$ is μ - Hölder continuous on $I\times\Omega\times[-\rho,\rho]$ uniformly with respect to ζ,η in bounded subsets of R^{n+1} and it is C^1 with respect to the remaining variables.

iii) There exists a function $K: R_+\to R_+$ and a constant $\epsilon\in(0,2]$ such that g satisfies the growth condition

$$|g(t,x,\xi,\zeta,\eta)| \le K(\rho)(1+|\eta|^{2-\epsilon})\text{for all}(t,x,\xi,\zeta,\eta)\in I\times\overline{\Omega}\times[-\rho,\rho]^2\times R^n.$$

iv) g is a monotone increasing function in the third variable.

v) The operator Λ is a continuous map from $C([-\gamma,0],C(\Omega))$ to $C(\Omega)$ satisfying

$$\|\Lambda\phi_1 - \Lambda\phi_2\|_{C(\Omega)} \le \|\phi_1-\phi_2\|^a_{C([-\gamma,0],C(\Omega))}\ \text{for some } a\in(0,1].$$

Further, $\Lambda\phi_1 \le \Lambda\phi_2$ whenever $\phi_1(t) \preceq \phi_2(t)$ for $t\in[-\gamma,0]$, where the symbol $\preceq$ denotes the natural order in the ordered Banach space $(C(\Omega),\preceq)$ defined by $\phi\preceq\psi$ if $\phi(x)\le\psi(x)$ for $x\in\Omega$.

To apply our abstract result to this problem we must define the operators A and f. Take $p>n$, let k_1 be a sufficiently large positive number, and define $X\equiv L_p(\Omega)$. Then the operator $A(t)$ is defined by

$$A(t)\phi = (\text{A}''(\cdot,t,D)+k_1)\phi,\quad \phi\in D(A(t))$$

where

$$D(A(t)) = \{\phi\in W^2_p(\Omega) : (\text{A}''(\cdot,t,D)+k_1)\phi\in L_p(\Omega) \text{ and } \text{B}''(\cdot,D)\phi = 0\}.$$

The family of operators $\{A(t) : t \in I\}$ as defined above satisfies assumptions (a)–(c) (see Friedmann [5]). Furthermore, by virtue of L_p-estimates for elliptic operators, there exist positive constants c_1 and c_2 such that

$$c_1 \|\phi\|_{W_p^2} \leq \|A(t)\phi\|_{L_p} \leq c_2 \|\phi\|_{W_p^2} \text{ for } t \in J \text{ and } \phi \in D(A(t)).$$

Fix a number α satisfying

$$\frac{1}{2} + \frac{n}{2p} < \alpha < (\frac{1}{2} - \epsilon).$$

The unknown solution $u(t, \cdot) = u(t)$ is considered as a curve in some appropriate space X of functions on W. As such it is to satisfy a corresponding initial value problem (2.1). In case of the Navier-Stokes system, the estimate

$$f(t, x_t, x) = g(\cdot, t, Ax_t, x(\cdot, t), \nabla x(\cdot, t)) + k_1 x(\cdot, t)$$

on the derivative of f does not influence the quadratic nonlinearity $-P[(x \cdot \nabla)x]$. However, it requires additional regularity of the external force $v \in L^2(J, V)$. Let $B : V \to R^n$ with $V \subset J$ be a linear operator such that there exists an invertible operator W^{-1} defined on $X \setminus kerW$ where W is defined by (3.1) for $v \in L^2(J, V)$. Under this construction the system (4.1)–(4.2) is an abstract formulation of system (2.3)–(2.4) (see [7]), hence it is null controllable on J by Theorem 3.1.

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