LIBERTAS MATHEMATICA, VOL XXI (2001)

HYPERBOLIC DIFFERENTIAL INEQUALITIES

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2000 Mathematical Subject Classification Code: 47H10, 47J20, 35B05, 35R10, 35L70.
Key words: weakly Picard operators, Gronwall's lemmas, hyperbolic inequalities.

1 Introduction

This paper presents certain generalization for hyperbolic differential inequalities of Gronwall-Wendorff's classical inequalities. The principal results are based on a operatorial inequality for weakly Picard operators (Rus [10]) and in finaly these involves Riemann function for a linear hyperbolic operator.

These inequalities can be applied to divers qualitative problems for PDE of hyperbolic type.

2 Weakly Picard operators

Definition 2.1. ([9], [10]) *Let (X,d) be a metric space. An operator $f : X \to X$ is Picard operator if there exists $x^* \in X$ such that*

(*i*) $F_f = \{x^*\}$
(*ii*) $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^*, *for all* $x_0 \in X$.

Definition 2.2. ([9], [10]) *Let (X,d) be a metric space. An operator $f : X \to X$ is weakly Picard operator if the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of f.*

Lemma 2.1 (Abstract Gronwall lemma; [10]). *Let (X,d) be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(*i*) *A is a Picard operator*
(*ii*) *A is monotone increasing.*
If x_A^* is the fixed point of the operator A, then
a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$
b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

Lemma 2.2. ([10]) *Let $(X,d,\leq)$ be an ordered metric space, $A : X \to X$ an operator and $x, y \in X$ such that*

$$x < y, \quad x \leq A(x), \quad y \geq A(y).$$

We suppose that

(i) *A is weakly Picard operator,*
(ii) *A is monotone increasing.*

Then

$a)$ $x \leq A^\infty(x) \leq A^\infty(y) \leq y$
$b)$ *$A^\infty(x)$ is the minimal fixed point of A in $[x,y]$ and $A^\infty(y)$ is the maximal fixed point of A in $[x,y]$.*

Lemma 2.3. ([10]) *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ be such that*

(i) $A \leq B \leq C$
(ii) *the operators A, B, C are WPOs*
(iii) *the operator B is monotone increasing.*

Then

$$x \leq y \leq z \;\Rightarrow\; A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

3 Gronwall type inequalities

Theory of differential and integral inequalities play a prominent role in the study of qualitative behavior of differential equations, nonlinear differential systems and in PDE.

We consider the following hyperbolic inequality

$$\frac{\partial^2 u}{\partial x \partial y} \leq f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad (x,y) \in \overline{D}, \tag{3.1}$$

and the Darboux problem

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad (x,y) \in \overline{D} \tag{3.2}$$

$$\begin{cases} u(x,0) = \varphi(x),\ x \in [0,a] \\ u(0,y) = \phi(y),\ y \in [0,b],\ \varphi(0) = \psi(0) \end{cases} \tag{3.3}$$

where $\overline{D} = [0,a] \times [0,b]$, $f \in C(\overline{D} \times \mathbb{R}^3)$, $\varphi \in C^1[0,a]$, $\psi \in C^1[0,b]$, $u \in C^1(\overline{D})$ and $\dfrac{\partial^2 u}{\partial x \partial y} \in C(\overline{D})$.

We have

Theorem 3.1. *If*

(i) $f \in C(\overline{D} \times \mathbb{R}^3)$,
(ii) $|f(x,y,u_1,u_2,u_3) - f(x,y,v_1,v_2,v_3)| \leq L_f \max(|u_i - v_i|), i = 1,2,3)$,
(iii) $\varphi \in C^1[0,a]$, $\psi \in C^1[0,b]$,
(iv) $f(x,y,\ldots) : \mathbb{R}^3 \to \mathbb{R}$ monotone increasing.

Then

a) the Darboux problem (2)+(3) has a unique solution u^*

b) if u *is a solution of (1)+(3) then* $u \leq u^*$.

PROOF. We put the problem (2)+(3) as a fixed point problem. If u is a solution of the problem (2)+(3), the $\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ is a solution of the following system

$$\begin{cases} u(x,y) = \varphi(x) + \psi(y) - \varphi(0) + \int\limits_0^x \int\limits_0^y f(s,t,u(s,t),v(s,t),w(s,t))dsdt \\ v(x,y) = \varphi'(x) + \int\limits_0^y f(x,t,u(x,t),v(x,t),w(x,t))dt \\ w(x,y) = \psi'(y) + \int\limits_0^x f(s,y,u(s,y),v(s,y),w(s,y))ds, \end{cases} \tag{3.4}$$

or in general form

$$\begin{aligned} u(x,y) &= A_1(u,v,w)(x,y) \\ v(x,y) &= A_2(u,v,w)(x,y) \\ w(x,y) &= A_3(u,v,w)(x,y), \end{aligned}$$

$u, v, w \in C(\overline{D})$.

If $(u,v,w) \in C(\overline{D})^3$ is a solution of (4) then $u \in C^1(\overline{D})$ and $v = \frac{\partial u}{\partial x}$, $w = \frac{\partial u}{\partial y}$ i.e., u is a solution of (2)+(3).

Let $X := C(\overline{D}) \times C(\overline{D}) \times C(\overline{D})$ and

$$\|(u,v,w)\| := \max\left(\max_{\overline{D}} |u(x,y)|e^{-\tau(x+y)}, \max_{\overline{D}} |v(x,y)|e^{-\tau(x+y)}, \max_{\overline{D}} |w(x,y)|e^{-\tau(x+y)}\right)$$

$(C(\overline{D}, +, \mathbb{R}, \|\cdot\|_B)$ is a Banach space.

Let $A : X \to X$, $(u,v,w) \to (A_1(u,v,w), A_2(u,v,w), A_3(u,v,w))$, we have

$$\|A(u_1,v_1,w_1) - A(u_2,v_2,w_2)\|_B \leq \frac{L_f}{\tau}\|(u_1,v_1,w_1) - (u_2,v_2,w_2)\|_B.$$

Thus if $\tau > 0$ is such that $\frac{L_f}{\tau} < 1$, then the operator A is a contraction so A is a Picard operator. From (iv) we have that A is monotone increasing. Let u be a solution of (1). Then

$$\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \leq A\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

From Lemma 2.1 we have that

$$\begin{aligned} u &\leq u^* \\ \frac{\partial u}{\partial x} &\leq \frac{\partial u^*}{\partial x} \\ \frac{\partial u}{\partial y} &\leq \frac{\partial u^*}{\partial y}. \end{aligned}$$

Example 3.1. (see [3], [7]) Let $a, b > 0$ and $\overline{D} = [0, a] \times [0, b]$. Let p, q, r and $g \in C(\overline{D})$. We consider the following hyperbolic inequality

$$\frac{\partial^2 u}{\partial x \partial y} + p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} + r(x,y)u \leq g(x,y), \quad (x,y) \in \overline{D} \tag{3.1'}$$

and the Darboux problem

$$\frac{\partial^2 u}{\partial x \partial y} + p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} + r(x,y)u = g(x,y), \quad (x,y) \in \overline{D} \tag{3.2'}$$

$$\begin{cases} u(x,0) = \varphi(x),\ x \in [0,a] \\ u(0,y) = \psi(y),\ y \in [0,b] \end{cases}, \quad \varphi(0) = \psi(0) \tag{3.2'}$$

where $\varphi \in C^1[0,a]$ and $\psi \in C^1[0,b]$.

We suppose that $p \leq 0$, $q \leq 0$ and $r \leq 0$.

Then the Darboux problem $(2') + (3')$ has a unique solution u^*. If u is a solution of $(1') + (3')$ then $u \leq u^*$. In this case

$$u^*(x,y) = v(0,0;x,y)\varphi(0) + \int_0^x v(s,0;x,y)(\varphi'(s) + b(s,0)\varphi(s))ds +$$

$$+ \int_0^y v(0,t;x,y)(\psi'(t) + a(0,t)\psi(t))dt + \iint_{\overline{D}} v(s,t;x,y)f(s,t)dsdt$$

where v is the Riemann function.

Remark. In the case in which the Riemann function of the operator

$$L(u) := \frac{\partial^2 u}{\partial x \partial y} + p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} + r(x,y)u$$

is positive (see [6]) we can study, in the same way, the inequality

$$L(u) \leq f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

4 Comparison theorems

We have

Theorem 4.1. *Let f_i, φ_i, ψ_i be as in the Theorem 3.1. If*

$$f_1 \leq f_2, \quad \varphi_1^{(k)} \leq \varphi_2^{(k)}, \quad \psi_1^{(k)} \leq \psi_2^{(k)}, \quad k = 0, 1,$$

then

$$u^*(\cdot, f_1, \varphi_1, \psi_1) \leq u^*(\cdot, f_2, \varphi_2, \psi_2).$$

PROOF. We consider the operator

$$A : (C(\overline{D})_0)^3 \to (C(\overline{D})_0)^3$$

defined by

$$(u, v, w) \mapsto (A_1(u, v, w), A_2(u, v, w), A_3(u, v, w))$$

where

$$C(\overline{D})_0 := \{u \in C(\overline{D}) |\ u(0, 0) = 0\}$$

and

$$A_1(u, v, w)(x, y) := u(x, 0) + v(0, y) + \int_0^x \int_0^y f_1(s, t, u(s, t), v(s, t), w(s, t))dsdt,$$

$$A_2(u, v, w)(x, y) := v(x, 0) + \int_0^y f_1(x, t, u(x, t), v(x, t), w(x, t))dt,$$

$$A_3(u, v, w)(x, y) := w(0, y) + \int_0^x f_1(s, y, u(s, y), v(s, y), w(s, y))ds.$$

At first we shall prove that the operator A is WPO.

Let

$$C(\overline{D})_{\varphi,\psi} := \{u \in C(\overline{D}_0 |\ u(x, 0) = \varphi(x),\ u(0, y) = \psi(y)\}$$

$$C(\overline{D})_{\varphi} := \{u \in C(\overline{D})_0 |\ u(x, 0) = \varphi(x)\}, \quad \varphi \in C[0, a]$$

$$C(\overline{D})^{\psi} = \{u \in C(\overline{D})_0 |\ u(0, y) = \psi(y)\}, \quad \psi \in C[0, b].$$

It is clear that

$$(C(\overline{D})_0)^3 = \left(\bigcup_{\varphi,\psi} C(\overline{D})_{\varphi,\psi}\right) \times \left(\bigcup_{\varphi} C(\overline{D})_{\varphi}\right) \times \left(\bigcup C(\overline{D})^{\psi}\right) =$$

$$= \bigcup_{\varphi,\psi,\theta,\chi} (C(\overline{D})_{\varphi,\psi} \times C(\overline{D})_{\theta} \times C(\overline{D})^{\chi})$$

is a partition of $(C(\overline{D})_0)^3$ and

$$X_{\varphi,\psi,\theta,\chi} := C(\overline{D})_{\varphi,\psi} \times C(\overline{D})_{\theta} \times C(\overline{D})^{\chi} \in I(A).$$

Moreover the operator $A|_{X_{\varphi,\psi,\theta,\chi}}$ is a PO (see the proof of the Theorem 3.1). So, the operator A is WPO and is monotone increasing. If $\varphi \in C[0, a]$, then we denote by $\widetilde{\varphi}$ the function $u \in C(\overline{D})$ defined by $u(x, y) = \varphi(x)$. Similarly to f_2 we have the operator B. It is clear that

$$A^{\infty}(\varphi_1 + \psi_1, \widetilde{\varphi}_1', \widetilde{\psi}_1') = u^*(\cdot, f_1, \varphi_1, \psi_1)$$

and

$$B^{\infty}(\varphi_2 + \psi_2, \widetilde{\varphi}_2', \widetilde{\psi}_2') = u^*(\cdot, f_2, \varphi_2, \psi_2).$$

Now, the theorem follows from the Lemma 2.3.

Remark. In [12] on considers the Darboux problem for the following hyperbolic partial differential system
$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u), \quad (x,y) \in \overline{D}.$$

If we consider the operator

$$A : C(\overline{D}, R^m)_0 \to C(\overline{D}, R^m)_0$$

defined by
$$A(u)(x,y) := u(x,0) + u(0,y) + \int_0^x \int_0^y f(t,s,u(t,s))dtds$$

then, in the conditions from [12], we have that the operator A is WPO. So, from Lemma 2.1 and Lemma 2.3 we have the results given in [12]. We have the similar considerations from some results given in [13], [11], [4] and [5].

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