

VARIATIONAL PRINCIPLES ON SEMI-METRIC STRUCTURES

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Abstract

A class of variational principles (including the well known Ekeland's one) is considered, over semi-metric structures.

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1. Introduction

Recently, in their 1995 paper, Gajek and Zagrodny [6] established a lot of mean value theorems which involves in a more efficient way the geometry of the codomain space. The core of their developments is a variational principle (cf. Theorem 2.1 in the quoted paper) which includes the well known Ekeland's [4]. But, the argument used in the proof of this principle (based, essentially, on their 1994 Weierstrass type statement [5]) is rather involved. So, we may ask of whether or not is this removable. It is our aim in the following to show that a positive answer to the posed question is possible; this will be the content of Section 4 below. The basic tool of our device is represented by a lot of Zorn type maximality results over semi-metric structures, to be found in Section 3. Note that, as another by-product of these, we get the monotone version of the variational Ekeland's principle obtained in Turinici [12]. Finally, Section 2 (with a preliminary character) collects some maximality principles over pseudometric structures. A number of further developments of these facts will be performed in a subsequent work.

2. Pseudometric maximality principles

Let X be a nonempty set and (\leq) , some *relation* over it. We shall term this object, *quasi-order* when it is reflexive and transitive; and *order*, provided it is, in addition, anti-symmetric. Further, let $d : X^2 \rightarrow [0, \infty]$ be a map. We call it *pseudometric*, when

(2D1) d is diagonal ($d(x, x) = 0, x \in X$).

Take in the following some quasi-order (\leq) and some pseudometric d over X . Call the point $z \in X$, (\leq, d)-maximal, if

(2D2) $z \leq u \leq v \implies d(u, v) = 0$.

In this context, we shall say that (\leq) is a *Zorn quasi-order*, provided

(2D3) $\left\{ \begin{array}{l} \text{for each } x \in X \text{ there exists a } (\leq, d)\text{-maximal} \\ \text{element } z \in X \text{ with } x \leq z. \end{array} \right.$

It is our aim in the following to establish sufficient conditions about these data so that such a property be retainable. To do this, we shall need some conventions and auxiliary facts.

Call the sequence (x_n) in X , *asymptotic* if

(2D4) $d(x_n, x_{n+1}) \longrightarrow 0$ as $n \rightarrow \infty$,

and *Cauchy*, provided

(2D5) $\left\{ \begin{array}{l} \text{for each } \varepsilon > 0 \text{ there exists a rank } n(\varepsilon) \\ \text{such that } n(\varepsilon) \leq n \leq m \implies d(x_n, x_m) < \varepsilon. \end{array} \right.$

Note that, for an individual sequence, we have a generic relation like

(2.1) *Cauchy property* \implies *asymptotic property*.

This, in particular, will be true for an ascending/descending sequence (in X). Moreover, from a global viewpoint, the reverse implication is also retainable: precisely, the global conditions

(2H1) *each ascending sequence is asymptotic*

and

(2H2) *each ascending sequence is Cauchy*

are equivalent each other. In fact, assume the ascending sequence (x_n) is not endowed with the Cauchy property. Then, an $\varepsilon > 0$ may be found so that

(2.2) *for each n , there exist p, q with $n \leq p \leq q$ and $d(x_p, x_q) \geq \varepsilon$.*

This immediately yields an ascending subsequence (y_n) of (x_n) with

(2.3) $d(y_{2n+1}, y_{2n+2}) \geq \varepsilon$, for all n ,

in contradiction to (2H1); hence the assertion. In this case, it will be meaningful calling this quasi-order (\leq), *regular* in case

(2D6) *one of the (mutually equivalent) conditions (2H1) or (2H2) is holding.*

Further, denote for each sequence (x_n) in X

(2D7) $\text{ubd}(x_n) = \bigcap_n X(x_n, \leq)$.

Here, by convention

(2D8) $X(u, \leq) = \{x \in X; u \leq x\}, u \in X$.

Each element of this set will be referred to as an *upper bound* of (x_n) ; and, if $\text{ubd}(x_n) \neq \emptyset$ we shall say that this sequence is *bounded from above*. Finally, call the ambient quasi-order (\leq), *normal* when

(2D9) *each ascending sequence is bounded from above.*

We are now in position to give an appropriate answer to the posed question.

PROPOSITION 2.1. *The generic implication below is valid*

(2.4) *regular & normal \implies Zorn.*

That is: each regular and normal quasi-order is necessarily a Zorn one.

Proof. Let the premises in this relation be in use. We claim that

(2.5) | for each $x \in X, \varepsilon > 0$, there exists $y = y(x, \varepsilon) \geq x$
 such that $y \leq u \leq v \implies d(u, v) < \varepsilon$.

Indeed, assume this would be false; that is, for some $x \in X, \varepsilon > 0$,

(2H3) for each $y \geq x$ there exist $u, v \in X$ with $y \leq u \leq v, d(u, v) \geq \varepsilon$.

Then, an ascending sequence (y_n) may be found so that (2.3) be retainable. But then, the regularity of (\leq) would be contradicted; hence the claim. It results that, for the given $x \in X$, an ascending Cauchy sequence (x_n) with $x_0 = x$ may be determined such that, for all n ,

(2.6) $x_n \leq u \leq v \implies d(u, v) < 2^{-n}$.

Let z be an upper bound of (x_n) (existing by the normality assumption). We intend to show that this is our desired element. Firstly, it is clear that $x \leq z$. Secondly, take $u, v \in X$ in accordance with $z \leq u \leq v$. By (2.6), one gets $d(u, v) = 0$; wherefrom, z is (\leq, d) -maximal. The proof is thereby complete. ■

An interesting unilateral counterpart of this result may be constructed under the lines below. Call the point $z \in X$, semi (\leq, d) -maximal, provided

(2D11) $z \leq w \implies d(z, w) = 0$.

In this context, we shall say that (\leq) is a semi Zorn quasi-order, if

(2D11) | for each $x \in X$ there exists a semi (\leq, d) -maximal
 element $z \in X$ with $x \leq z$.

As before, we shall be interested in deriving sufficient conditions under which the introduced property be retainable. To do this, we start from the generic implication

(2.7) (\leq, d) -maximal \implies semi (\leq, d) -maximal;

so, as a consequence of this,

(2.8) Zorn property \implies semi Zorn property.

In other words, the result above may be also viewed as a (partial) answer to the posed question. For a technical completion of it, we need some conventions. Call the sequence (x_n) in X , semi-asymptotic, in case

(2D12) $\liminf_n d(x_n, x_{n+1}) = 0$;

and semi-Cauchy, provided

(2D11) | for each $\varepsilon > 0$ there exists a rank $n = n(\varepsilon)$
 such that $p \geq n \implies d(x_n, x_p) < \varepsilon$.

Note that, for an individual sequence, we have a generic implication like

(2.9) semi-Cauchy property \implies semi-asymptotic property.

This, in particular, would be valid for each ascending/descending sequence in X . Moreover, from a global viewpoint, the reverse implication is also retainable; precisely, the global conditions

(2H4) each ascending sequence is semi-asymptotic

and

(2H5) each ascending sequence is semi-Cauchy

are equivalent each other. (The proof is very similar to the one involving (2H1) and (2H2); so, we omit it). As before, it will be meaningful calling the ambient quasi-order (\leq) , semi-regular, provided

(2D14) one of the (mutually equivalent) conditions (2H4) or (2H5) is holding.

The announced statement may now be formulated as follows.

PROPOSITION 2.2. *We have the generic implication*

(2.10) *semi-regular & normal \implies semi Zorn.*

That is: each semi-regular and normal quasi-order is a semi Zorn one.

Proof. Let the premise of this implication be in use. We claim that

(2.11) $\left| \begin{array}{l} \text{for each } x \in X, \varepsilon > 0 \text{ there exists } y = y(x, \varepsilon) \geq x \\ \text{such that } y \leq u \implies d(y, u) < \varepsilon. \end{array} \right.$

In fact, assume this would be false; that is, for some $x \in X, \varepsilon > 0$,

(2H6) *for each $y \geq x$ there exists $u \in X$ with $y \leq u, d(y, u) \geq \varepsilon$.*

Then, an ascending sequence (y_n) may be found so that

(2.12) *$d(y_n, y_{n+1}) \geq \varepsilon$, for all n .*

But, in such a case, the semi-regularity of (\leq) will be contradicted; hence the claim. It results from this that, for the given $x \in X$, an ascending semi-Cauchy sequence (x_n) with $x_0 = x$ may be constructed so that, for all n ,

(2.13) *$x_n \leq u \implies d(x_n, u) < 2^{-n}$.*

Let z be any upper bound of (x_n) (existing by the normality condition we adopted). It is clear that this is a semi (\leq, d) -maximal element of X which majorizes x . Hence the conclusion. ■

Now, by the generic implication

(2.14) *regular \implies semi-regular*

the premise of (2.10) is weaker than the one of (2.4). On the other hand, by the remark (2.8), it is also clear that the conclusion of Proposition 2.2 is weaker than the one stated in Proposition 2.1. A natural question is that of determining sufficient conditions upon our data under which the premise of Proposition 2.2 should give the conclusion of Proposition 2.1. To this end, let us define a convergence structure over X , by the convention

(2D15) $\left| \begin{array}{l} x_n \rightarrow x \text{ if for each } \varepsilon > 0 \text{ there exists} \\ \text{a rank } n(\varepsilon) \text{ such that } n \geq n(\varepsilon) \implies d(x_n, x) < \varepsilon. \end{array} \right.$

Remark at this moment that, by the general working context, no connection between the (semi-) Cauchy property and this convergence property is to be made. Put also

(2D16) *$x_n \uparrow x$ iff $x_n \rightarrow x$ and $x_n \leq x$, for all n .*

In this case, x will be referred to as a limit upper bound of (x_n) ; and the set of all these will be denoted $\text{lub}(x_n)$.

PROPOSITION 2.3. *Suppose that the premise of (2.10) (cf. Proposition 2.2) is true, as well as*

(2H7) $\left| \begin{array}{l} \text{for each ascending semi-Cauchy sequence } (x_n) \text{ with } \text{lub}(x_n) \neq \emptyset \\ \text{one has } u, v \in \text{lub}(x_n), u \leq v \implies d(u, v) = 0. \end{array} \right.$

Then, conclusion of Proposition 2.1 is retainable.

Proof. Given the arbitrary fixed $x \in X$, let (x_n) be the ascending semi-Cauchy sequence (with $x_0 = x$) produced in the proof of the quoted statement. Note that

$$\emptyset(x_n) \subseteq \text{lub}(x_n) \quad (\text{hence } \text{lub}(x_n) \neq \emptyset).$$

Fix $z \in \text{ubd}(x_n)$. If $u, v \in X$ satisfy $z \leq u \leq v$ then $x_n \uparrow u, x_n \uparrow v$ (by the property (2.13));so, $d(u, v) = 0$ (if we take (2H7) into account). Hence the conclusion. ■

The following technical aspect is to be noted. Call the pseudometric d over X , (\leq) -triangular, when

$$(2D17) \quad \left| \begin{array}{l} \text{for each } \varepsilon > 0 \text{ there exists } \delta = \delta(\varepsilon) > 0 \text{ such that} \\ x \leq y \leq z, d(x, y), d(x, z) \leq \delta \implies d(y, z) \leq \varepsilon. \end{array} \right.$$

It is not hard to see that,under

$$(2H8) \quad d \text{ is } (\leq)\text{-triangular,}$$

each of the above introduced notions is equivalent with its corresponding semi-notion. Hence, in particular,

$$(2.15) \quad \text{regular} \iff \text{semi-regular (under (2H8))}.$$

This yields the following practical statement. (As before, (2H8) prevails).

PROPOSITION 2.4. *Suppose that the quasi-order (\leq) is semi-regular (or, equivalently, regular). Then, it is necessarily a Zorn (or, equivalently, a semi Zorn) one.*

Some remarks are in order. The results above are being obtained under the line in Turinici [11]. As precised in the above quoted paper, these are compatible with the contributions in this area due to Altman [1], Brezis and Browder [2] or Kang and Park [8]. And technically speaking, such facts may be viewed as denumerable versions of the Zorn Maximality Principle. A quasi-uniform version of these statements is directly available; but,this is not essential to the developments below. For a number of topological aspects we refer to Brunner [3] and the references therein.

3. Main results

Let the working conventions be in force. It is our aim in the following to indicate a basic application of the obtained facts. (Its usefulness will become clear in the next section).The most appropriate setting to be considered is the semi-metric one. But,we must say that an extension of these developments to pseudometric structures is almost immediate. Some other aspects will be discussed elsewhere.

Let M be a nonempty set and (\leq) , some quasi-order over it. As usually, we shall identify (\leq) with its graph $(= \{(x, y) \in M^2; x \leq y\})$. Further, let $d : M^2 \rightarrow [0, \infty[$ be a semi-metric over M . Note that, in particular, d is triangular (hence (\leq) -triangular). So, all notions introduced in the preceding section are equivalent with their corresponding semi-notions. In addition,we have the generic (sequence-type) implication

$$(3.1) \quad \text{convergent} \implies \text{Cauchy}.$$

When the reciprocal of this also holds, the space (M, d) is called complete. As a variant of this, call the ambient structure, (\leq) -complete provided

$$(3D1) \quad \text{each ascending Cauchy sequence converges.}$$

It is trivial that each complete quasi-ordered structure is (\leq) -complete too; but the converse implication is not true,in general. The introduced concept suggests us considering similar ones (related to (\leq)). Precisely, call the part Z of M , (\leq) -closed, if

(3D2) the *limit of each ascending sequence in Z belongs to Z .*

As before, one has the generic relation: closed \implies (\leq) -closed; the reciprocal is not generally valid. In this context, given another quasi-order \preceq over M , we shall say that (\leq) is \preceq -closed, in case

(3D3) $M(x, \leq)$ is (\preceq) -closed, for each $x \in M$.

In particular, when (\preceq) is just (\leq) , we shall refer to this as (\leq) being self-closed; that is (equivalently)

(3D4) the *limit of each ascending sequence in M is an upper bound of it.*

And, if $(\preceq) = M^2$ (the trivial quasi-order on M), the same property will be qualified as (\leq) being closed. For a number of related concepts we refer to Nachbin [10, ch.1, Section 1].

Having these precised, let (\leq_1) and (\leq_2) be a couple of quasi-orders over M . Put

(3D5) $(\leq) = (\leq_1) \cap (\leq_2)$.

This is again a quasi-order over M , referred to as the product between the considered quasi-orders. It will be of interest for us to determine sufficient conditions involving these objects under which (\leq) be a Zorn quasi-order. In this direction, our main result is

THEOREM 3.1. *Suppose that*

(3H1) *either (\leq_1) or (\leq_2) is regular*

(3H2) *(M, d) is (\leq) -complete*

(3H3) *both (\leq_1) and (\leq_2) are (\leq) -closed.*

Then, (\leq) is a Zorn quasi-order, in the sense: for each $u \in M$, there exists $v \in M$ with
(a) $u \leq v$, (b) $v \leq w \implies d(v, w) = 0$.

Proof. By (3H1), (\leq) is necessarily regular. We now intend to show that this (product) quasi-order is normal too. (And then, by Proposition 2.1, we are done). Let (x_n) be an ascending (modulo (\leq)) sequence in M . By (3H2), $x_n \rightarrow x$ as $n \rightarrow \infty$, for some $x \in M$. And this, in combination with (3H3), yields

$$x \in M(x_n, \leq_1) \cap M(x_n, \leq_2), \text{ for all } n;$$

or, equivalently, x is an upper bound (modulo (\leq)) of (x_n) . The proof is thereby complete. ■

A basic particular case of these developments is to be determined under the lines below. Let $g : M \rightarrow R$ be a function and (\preceq) , some quasi-order over M . Call this function, (\preceq) -lsc, provided

(3D6) $\{x \in M; g(x) \leq \lambda\}$ is (\preceq) -closed, for each λ .

Note that, if (\preceq) coincides with the trivial quasi-order over M , this property is nothing but the familiar lsc one. The relationship with the previous one are expressed as

(3.2) $\text{lsc} \implies (\preceq)\text{-lsc}$ (for all such (\preceq));

but, the converse implication is not true, in general. Another useful choice for (\preceq) is

$(\preceq) = (\beta, g)$ (with g taken as before).

Here, for each $\beta \geq 0$, (β, g) stands for the quasi-order over M described as

(3D7) $x(\beta, g)y$ iff $\beta d(x, y) \leq g(x) - g(y)$.

The corresponding (3D6) property for $\beta = 0$ will be referred to as g being self-lsc; that is,

(3D8) $\left\{ \begin{array}{l} \text{for each sequence } (x_n) \text{ in } M \text{ and each } x \in M \text{ with } (g(x_n)) \\ \text{decreasing and } x_n \rightarrow x \text{ one has } g(x_n) \geq g(x), \text{ for all } n. \end{array} \right.$
 We are now in position to state

THEOREM 3.2. *Let the quasi-order (\leq) over M and the function $g : M \rightarrow R$ be such that*

- (3H4) *g is bounded from below*
- (3H5) *(M, d) is $(\leq) \cap (0, g)$ -complete*
- (3H6) *(\leq) is $(\leq) \cap (0, g)$ -closed*
- (3H7) *g is $(\leq) \cap (0, g)$ -lsc.*

Then, for each $\beta \geq 0$ and each $u \in M$, there exists $v = v(\beta, u) \in M$ with the properties

- (3.3) *$u \leq v$ and $\beta d(u, v) \leq g(u) - g(v)$*
- (3.4) *for each $w \in M$ with $v \leq w, d(v, w) > 0$ we have $\beta d(v, w) > g(v) - g(w)$*
- (3.5) $\left\{ \begin{array}{l} \text{for each } w \in M \text{ with } \beta d(v, w) \leq g(v) - g(w), d(v, w) > 0, \\ \text{we cannot have a relation like } v \leq w. \end{array} \right.$

Proof. We shall verify that the conditions of the quoted statement are fulfilled, with $(\leq_1) = (\leq)$, $(\leq_2) = (\beta, g)$. Firstly, the very definition of (β, g) tells us (along with the boundedness condition (3H4)) that (β, g) is a regular quasi-order (on M); i.e., (3H1) holds (in this setting). Secondly, in view of (3H5), plus

$$(3.6) \quad (\leq) \cap (\beta, g) \subseteq (\leq) \cap (0, g)$$

it is clear that (3H2) holds too. Finally, the same inclusion shows that (\leq) is $(\leq) \cap (\beta, g)$ -closed as well as (by (3H7)) that g is $(\leq) \cap (\beta, g)$ -lsc; wherefrom, (3H3) is directly obtainable (via standard arguments). This proves our initial claim about the considered couple of quasi-orders. But then, from the conclusions in Theorem 3.1, there exists, for the starting point $u \in M$, some $((\leq) \cap (\beta, g), d)$ -maximal element v in M with

$$u((\leq) \cap (\beta, g))v \text{ (hence } u \leq v, \beta d(u, v) \leq g(u) - g(v)).$$

It is now clear that the obtained element has all the properties we need. Hence the conclusion. ■

Technically speaking, the regularity conditions (3H5)–(3H7) are, in general, difficult to be checked; because these are formulated in terms of the product quasi-order $(\leq) \cap (\beta, g)$. So, it is natural asking of to what extent can we express them in terms of (\leq) only. An appropriate – and useful – answer is contained in

THEOREM 3.3. *Assume (3H4) is in use, as well as*

- (3H8) *(M, d) is (\leq) -complete*
- (3H9) *(\leq) is self-closed*
- (3H10) *g is (\leq) -lsc.*

Then, conclusions of Theorem 3.2 are to be retained (in this framework).

A direct verification of the claimed facts is to be derived from the very definition of the involved concepts; we do not give details. Note that, under the acceptance of (3H9), a sufficient condition for (3H10) is

$$(3H11) \quad g \text{ is } (\leq)\text{-decreasing.}$$

And then, the corresponding version of Theorem 3.3 is nothing but the monotone variational principle in Turinici [12] obtained via similar methods. On the other hand, another sufficient condition for (3H10) is

(3H12) *g is lsc.*

Hence, Theorem 3.3 also includes the well-known Ekeland's variational principle [4].

Now, a natural choice for the ambient quasi-order (\leq) is, evidently,

$(\leq) = (\alpha, f)$, for some $\alpha \geq 0$, $f : M \rightarrow R$.

In this case, the following completion of Theorem 3.2 above is retainable.

THEOREM 3.4. *Let the functions f, g from M to R be such that (3H4) is true, as well as*

(3H13) *(M, d) is $(0, f) \cap (0, g)$ -complete*

(3H14) *both f and g are $(0, f) \cap (0, g)$ -lsc.*

Then, for each couple $\alpha \geq 0, \beta > 0$, and each $u \in M$ there exists $v = v(\alpha, \beta, u)$ in M with the properties

(3.7) $\alpha d(u, v) \leq f(u) - f(v), \beta d(u, v) \leq g(u) - g(v)$

(3.8) $\left\{ \begin{array}{l} \text{for each } w \in M \text{ with } \alpha d(v, w) \leq f(v) - f(w), d(v, w) > 0, \\ \text{one has } \beta d(v, w) > g(v) - g(w). \end{array} \right.$

(3.9) $\left\{ \begin{array}{l} \text{for each } w \in M \text{ with } \beta d(v, w) \leq g(v) - g(w), d(v, w) > 0, \\ \text{we must have a relation like } \alpha d(v, w) > f(v) - f(w). \end{array} \right.$

Note that a natural sufficient condition for (3H13) to be true is

(3H15) *(M, d) is complete.*

And, a sufficient condition for (3H14) is

(3H16) *both f and g are self-lsc over M .*

Clearly, a symmetric result with respect to the preceding one is deducible by simply interchanging (α, f) with (β, g) ; we do not give details. As already precised, such statements include, among others, the variational Ekeland's principle [op.cit.]. So, it would be interesting to clarify the logical effectiveness of this extension. For a number of related facts we refer to Manka [9] and the references therein.

4. Some particular aspects

The obtained statements are not only interesting by themselves. For, as we shall see, these are the starting point in deducing a lot of variational principles comparable with the ones in Gajek and Zagrodny [6] (with immediate applications to geometric mean value theorems). The natural setting to be considered is the metrical one; but, we must say that the pseudo-metric extension of these is immediate. Precisely, take a complete metric space (X, d) as well as a couple of functions f, g from X to $R \cup \{\infty\}$, with

(4H1) $D = \text{dom}(f) \cap \text{dom}(g) \neq \emptyset$

(4H2) *both f and g are $(0, f) \cap (0, g)$ -lsc over X .*

As a first application of the developments in the preceding section, we have

THEOREM 4.1. *Let the (nonempty) part E of D be such that*

(4H3) E is $(0, f) \cap (0, g)$ -complete

(4H4) g is bounded below on E .

Then, for each $\alpha > 0$ and each $u \in E$ there exists $v = v(\alpha, u)$ in E with

(4.1) $f(u) \geq f(v)$, $\alpha d(u, v) \leq g(u) - g(v)$

*(4.2) | for each w in E distinct from v and each $\lambda \geq \alpha$
with $\lambda d(v, w) \leq g(v) - g(w)$ one has $f(v) < f(w)$*

*(4.3) | for each w in E distinct from v fulfilling $f(v) \geq f(w)$
and each $\lambda \geq \alpha$ one has $\lambda d(v, w) > g(v) - g(w)$.*

Proof. By the admitted hypotheses, Theorem 3.4 applies with $M = E$ and $(\alpha, f; \beta, g)$ substituted by $(0, f; \alpha, g)$. So, for the starting point $u \in E$ there exists some point $v = v(\alpha, u)$ in E fulfilling (3.7) as well as (cf. the choice of our data)

(4.4) $w \in E$, $f(v) \geq f(w)$, $\alpha d(v, w) \leq g(v) - g(w) \implies v = w$.

Now, evidently, (3.7) \implies (4.1). So, it remains to prove that (4.4) gives both (4.1) and (4.2). Assume that the pair (w, λ) is as in the premise of (4.2), and the conclusion of the same were not true. Then, necessarily (as $\lambda \geq \alpha$)

$0 \cdot d(v, w) \leq f(v) - f(w)$, $v \neq w$, $\alpha d(v, w) \leq g(v) - g(w)$,

in contradiction to (4.4); hence (4.2) follows. Likewise, take w as in the first half of the premise in (4.3), as well as some $\lambda \geq \alpha$. If the conclusion in the second half of the same is not valid, then $\lambda d(v, w) \leq g(v) - g(w)$ (hence $\alpha d(v, w) \leq g(v) - g(w)$). And this, in combination with the choice of our data yields a contradiction to (4.4); so, (4.3) must be true. The proof is complete. ■

Now, the structural condition (4H3) is fulfilled as soon as

(4H5) $E = D(G)$, for some part G of X with $D \cap G \neq \emptyset$.

This results at once from the completeness of our ambient space and the regularity condition (4H2); we do not give details. On the other hand, the requirement (4H2) imposed upon (f, g) is clearly fulfilled when

(4H6) f is self-lsc and g is lsc over X .

Note that, in such a context, the corresponding version of Theorem 4.1 is comparable with the variational principle in Gajek and Zagrodny [op.cit., Th.2.1] (precisely, the case $\beta \geq 0$, in their notation). This "comparative" aspect refers to the fact that, in many concrete cases, the operational information contained in these results is, practically, the same; we shall discuss this elsewhere.

Now, a second application of the developments in the preceding section is the following variational type result. Let again (X, d) be a complete metric space and f, g two functions from X to $R \cup \{\infty\}$ fulfilling (4H1) as well as (for each $\lambda > 0$)

(4H7) both f and $g + \lambda f$ are $(0, f) \cap (0, g + \lambda f)$ -lsc.

THEOREM 4.2. *Let the nonempty part E of D be such that, for each $\lambda > 0$,*

(4H8) E is $(0, f) \cap (0, g + \lambda f)$ -complete

(4H9) $g + \lambda f$ is bounded below on E .

Then, for each triplet (α, β, γ) with $\alpha, \gamma > 0$, $\beta \geq 0$, and each $u \in E$, there exists $v = v(\alpha, \beta, \gamma, u)$ in E with

$$(4.5) \quad f(u) \geq f(v), \alpha d(u, v) \leq g(u) - g(v) + \frac{\alpha + \beta}{\gamma} (f(u) - f(v)),$$

for which the conclusions (4.2) and (4.3) remain valid; and, in addition,

$$(4.6) \quad \left\{ \begin{array}{l} \text{for each } w \in E \text{ distinct from } v \text{ and each } \nu \leq \beta \text{ with} \\ 0 \leq g(v) - g(w) + \nu d(v, w) \text{ one has } \gamma d(v, w) > f(v) - f(w). \end{array} \right.$$

Proof. By the hypotheses we adopted, Theorem 3.4 is again applicable with $M = E$ and $(\alpha, f; \beta, g)$ substituted by $(0, f; \alpha, g + \frac{\alpha + \beta}{\gamma} f)$. So, for the starting point $u \in E$ there exists some point $v = v(\alpha, \beta, \gamma, u)$ in E fulfilling (3.7), as well as (cf. the choice of our data)

$$(4.7) \quad \left\{ \begin{array}{l} \text{whenever } w \in E \text{ fulfils } f(v) \geq f(w) \text{ and} \\ \alpha d(v, w) \leq g(v) - g(w) + \frac{\alpha + \beta}{\gamma} (f(v) - f(w)) \text{ then } v = w. \end{array} \right.$$

Now, evidently

$$(4.7) \implies (4.4) \quad (\text{because } (0, f) \cap (0, g + \frac{\alpha + \beta}{\gamma} f) \supseteq (0, f) \cap (0, g)).$$

And, from this, conclusions (4.2)+(4.3) are clear (by the argument we already sketched in Theorem 4.1). It remains to establish that (4.6) holds too. Let $w \in E$ be as in the premise of this implication; i.e., $w \neq v$ and

$$(4H10) \quad 0 \leq g(v) - g(w) + \nu d(v, w) \quad (\leq g(v) - g(w) + \beta d(v, w))$$

Suppose that the conclusion of (4.6) would be false, in the sense

$$(4H11) \quad \gamma d(v, w) \leq f(v) - f(w) \quad (\text{hence } d(v, w) \leq \frac{1}{\gamma} (f(v) - f(w))).$$

Combining these gives $f(v) \geq f(w)$, as well as $0 \leq g(v) - g(w) + \frac{\beta}{\gamma} (f(v) - f(w))$; wherefrom (by (4H11) again) $\alpha d(v, w) \leq g(v) - g(w) + \frac{\alpha + \beta}{\gamma} (f(v) - f(w))$. In other words, the premise of (4.7) is realized. But then (cf. its conclusion) $v = w$, in contradiction to the accepted choice for w . Hence (4.6) is true, as claimed. The proof is complete.

Now, exactly as before, the structural condition (4H8) is fulfilled under (4H5). On the other hand, the requirement (4H7) imposed upon (f, g) is clearly fulfilled when (4H6) is admitted. (The proof is a direct consequence of the involved concepts; we do not give details). Note that, in such a case, the corresponding variant of Theorem 4.2 is comparable with the one in Gajek and Zagrodny [op.cit., Th. 2.1] (precisely, the case $\beta < 0$, in their notation). It also includes a number of related variational principles described in Hyers, Isac and Rassias [7, ch.5, Section 4]. This will be discussed in a subsequent paper.

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