

## GROUPS AND RELATED-SYSTEMS

STEFAN ANTOHE

### 1. Introduction

Given  $\rho$ , a binary relation defined on the set  $G$ ,  $\rho \subseteq G \times G$ , then  $(G, \rho)$  is called the *binary related-system* or *related-system* [3].

Let  $(G, \rho)$ ,  $(G', \rho')$  be the related-systems. A map  $\varphi: G \rightarrow G'$  is called *rel-morphism* if  $(x, y) \in \rho \Rightarrow (\varphi(x), \varphi(y)) \in \rho'$ , i.e.  $x\rho y \Rightarrow \varphi(x)\rho'\varphi(y)$ .

The class of related-systems together with the rel-morphisms form a category noted *Rel*. In particular, one obtains: the category *Pre.*, the category *Ord.*, the category *Graph* [1,3].

A multiplicative group  $G$ , with identity 1, that has a class of the related-system defined on the set  $G$ , is called *G-related system*. This paper presents a study of the operations with these related-systems, their categorical properties, and the problems of representation.

### 2. The Category of $G$ -related Systems

Let  $(G, \cdot)$  be a commutative group, with identity 1. If  $S \subseteq G$ , we denote  $S^{-1} = \{s^{-1} : s \in S\}$ . We shall introduce the following notions:

**Definition 2.1**

A set  $S \subseteq G$  is called *G-self-invertible set* of  $G$  if  $S = S^{-1}$ . We denote

$$\mathcal{S}(G) = \{S : S \subseteq G, S = S^{-1}\} \quad (2.1)$$

**Definition 2.2**

Let  $G$  be a group and  $S \in \mathcal{S}(G)$ . The related-system  $(G, \rho(G, S))$ , where  $\rho(G, S)$  is defined by

$$(x, y) \in \rho(G, S) \Leftrightarrow xy^{-1} \in S \quad (2.2)$$

is called *G-related-system*, briefly denoted by  $(G, \rho_S)$ . We denote by

$$\mathcal{R}(G) = \{(G, \rho(G, S)) : S \in \mathcal{S}(G)\} \quad (2.3)$$

**Remark 2.3**

Let  $G$  be a group,  $S \in \mathcal{S}(G)$  and  $(G, \rho(G, S)) \in \mathcal{R}(G)$ . The following statements are true:

- 1) The relation  $\rho(G, S)$  is symmetric;

- 2) For every  $x \in S$ ,  $(x, 1) \in \rho(G, S)$ ;
- 3) If  $1 \in S$ , then the relation  $\rho(G, S)$  is reflexive;
- 4) If  $1 \notin S$ , then  $(G, \rho(G, S))$  is a quasi-Cayley graph [2, 4];
- 5) If the set  $S$  is stable (i.e.  $x, y \in S \Rightarrow xy \in S$ ), then  $\rho(G, S)$  is a transitive relation;
- 6)  $\emptyset, G \in S(G)$ .

**Theorem (2.4)**

Let  $(G, \rho_S)$  be a  $G$ -related-system. The relation  $\rho_S$  is an equivalence relation if and only if the set  $S$  is a subgroup of  $G$ .

**Proof.**

Because  $\rho_S$  is reflexive, for every  $x \in S$ ,  $x\rho_S x$ , i.e.  $xx^{-1} \in S$ , hence  $1 \in S$ . If  $x, y \in S$ , then  $x\rho_S 1$  and  $1\rho_S y$  belongs to  $S$ , hence  $x\rho_S y$ . This results in  $xy^{-1} \in S$ , i.e.  $S$  is a subgroup of  $G$ .

The reciprocal is obvious.

**Theorem 2.5**

Let  $f : G \rightarrow G'$  be a group morphism,  $S \subseteq G$  and  $S' \subseteq G'$ . The following statements hold:

- 1) If  $S \in S(G)$ , then  $f(S) \in S(G')$ ;
- 2) If  $S' \in S(G')$  and  $f(S) \subseteq S'$ , then  $f$  is a rel-morphism;
- 3) The map  $1_G : G \rightarrow G$ ,  $1_G(x) = x$ , for every  $x \in G$ , is a rel-morphism.

**Proof. :**

1) Because  $S^{-1} = S$ , we have

$$(f(S))^{-1} = \{(f(s))^{-1} : s \in S\} = \{f(s^{-1}) : s \in S\} = f(S^{-1}) = f(S).$$

2) The following implications provide the proof:

$$(x, y) \in \rho(G, S) \Leftrightarrow xy^{-1} \in S \Rightarrow f(xy^{-1}) \in f(S) \Leftrightarrow f(x)(f(y))^{-1} \in f(S) \subset S' \Rightarrow \\ \Rightarrow (f(x), f(y)) \in \rho(G', S')$$

**Theorem 2.6**

The class of  $G$ -related systems is a subcategory of category *Rel.* with the direct products.

**Proof. :**

The class of all  $G$ -related systems together with the rel-morphisms forms a category; the composition of rel-morphisms is the usual composition of maps and the identity rel-morphism is the identity map.

Let  $(G_i, \rho(G_i, S_i))$ ,  $i \in I$ , be a family of  $G$ -related systems.

We denote by

$$G = \prod_{i \in I} G_i = \{(x_i)_{i \in I} : x_i \in G_i, i \in I\} \quad (2.4)$$

the direct product of groups  $G_i$ ,  $i \in I$  and by

$$S = \prod_{i \in I} S_i = \{(s_i)_{i \in I} : s_i \in S_i, i \in I\}.$$

Because of

$$S^{-1} = \left( \prod_{i \in I} S_i \right)^{-1} = \prod_{i \in I} S_i^{-1} = \prod_{i \in I} S_i,$$

we have  $\prod_{i \in I} S_i \in S\left(\prod_{i \in I} G_i\right)$ .

It is clear that the applications  $p_i : \prod_{i \in I} G_i \rightarrow G_i$ ,  $i \in I$ ,  $p_i(x_j)_{j \in I} = x_i$ , are rel-morphisms. It is easy to show that  $\left(\prod_{i \in I} G_i, \rho\left(\prod_{i \in I} G_i, \prod_{i \in I} S_i\right)\right)$  with the family of rel-morphisms  $(p_i)_{i \in I}$  is a direct product of a family  $(G_i, \rho(G_i, S_i))$ ,  $i \in I$ .

**Corollary (2.7)**

Let  $(G_i)_{i \in I}$  be a family of commutative groups and  $S_i \subseteq G_i$ ,  $i \in I$ . The following statements are true:

- 1) If  $S_i \in \mathcal{S}(G_i)$ ,  $i \in I$ ,  $\prod_{i \in I} S_i \in \mathcal{S}(\prod_{i \in I} G_i)$ ;
- 2) The related systems  $(\prod_{i \in I} G_i, \rho(\prod_{i \in I} G_i, \prod_{i \in I} S_i))$  and  $\prod_{i \in I} (G_i, \rho(G_i, S_i))$  are isomorphic.

### 3. Latticeal Properties. The Representation of G-related Systems

In the following part  $G$  is assumed to be a commutative group.

**Theorem (3.1)**

The set  $\mathcal{S}(G)$  of all self-invertible sets in  $G$  is a Boole algebra, complete with respect to intersection, union, and complementary of a set.

**Proof.** :

If we denote  $\mathcal{P}(G) = \{X : X \subseteq G\}$ , then  $(\mathcal{P}(G), \cap, \cup, ')$  is a Boole algebra and  $\mathcal{S}(G) \subset \mathcal{P}(G)$ . Hence, it is sufficient to show that  $\mathcal{S}(G)$  is a subalgebra of  $\mathcal{P}(G)$ .

Indeed, if  $S_1, S_2 \in \mathcal{S}(G)$ , we have:

$$(S_1 \cap S_2)^{-1} = S_1^{-1} \cap S_2^{-1}, \quad (S_1 \cup S_2)^{-1} = S_1^{-1} \cup S_2^{-1},$$

hence  $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{S}(G)$ . It results that  $(\mathcal{S}(G), \cap, \cup)$  is a distributive lattice.

The set  $\emptyset, G \in \mathcal{S}(G)$  hence  $\emptyset$  is a zero element and  $G$  is a unit element of  $\mathcal{S}(G)$ . Obviously, if  $S \in \mathcal{S}(G)$ , then  $S' = G \setminus S \in \mathcal{S}(G)$  is a Boole subalgebra of  $(\mathcal{P}(G), \cap, \cup, ')$ .

**Theorem (3.2)**

The sets  $\mathcal{R}(G)$  of  $G$ -related systems can be organized with a Boole algebra structure relative to the operations deduced from Boole algebra  $\mathcal{S}(G)$ .

**Proof.** :

We define the following operations:

$$(G, \rho(G, S_1)) \cap (G, \rho(G, S_2)) = (G, \rho(G, S_1 \cap S_2)) \tag{3.1}$$

$$(G, \rho(G, S_1)) \cup (G, \rho(G, S_2)) = (G, \rho(G, S_1 \cup S_2)) \tag{3.2}$$

$$(G, \rho(G, S))' = (G, \rho(G, G \setminus S)) \tag{3.3}$$

denoted with the same symbols.

It is easy to check that  $(\mathcal{R}(G), \cap, \cup)$  is a distributive lattice. In the lattice  $\mathcal{R}(G)$ ,  $(G, \rho(G, \emptyset))$  is a zero element and  $(G, \rho(G, G))$  is a unit element. If  $(G, \rho(G, S)) \in \mathcal{R}(G)$ , then  $(G, \rho(G, S')) = (G, \rho(G, G \setminus S))$  verifies:

$$(G, \rho(G, S)) \cap (G, \rho(G, S')) = (G, \rho(G, S \cap S')) = (G, \rho(G, \emptyset)),$$

$$(G, \rho(G, S)) \cup (G, \rho(G, S')) = (G, \rho(G, S \cup S')) = (G, \rho(G, G)),$$

hence  $(G, \rho(G, S'))$  is the complement of  $(G, \rho(G, S))$ .

**Remark (3.3)**

The operations (3.1), (3.2), and (3.3) are called intersection, union, and complementarity.

**Theorem (3.4)**

The Boole algebras  $\mathcal{S}(G)$  and  $\mathcal{R}(G)$  are isomorphic.

**Proof.** :

We consider the map  $\varphi : \mathcal{S}(G) \rightarrow \mathcal{R}(G)$ ,  $\varphi(S) = (G, \rho(G, S))$ . According to (3.1), (3.2), and (3.3) it results that  $\varphi$  verifies:

$$\varphi(S_1 \cap S_2) = \varphi(S_1) \cap \varphi(S_2) \quad (3.4)$$

$$\varphi(S_1 \cup S_2) = \varphi(S_1) \cup \varphi(S_2) \quad (3.5)$$

$$\varphi(S') = (\varphi(S))', \quad (3.6)$$

i.e.  $\varphi$  is a morphism of Boole algebras.

From definition it results that  $\varphi$  is surjective. If  $S_1 \neq S_2$ , there is an element  $s \in S_1 \setminus S_2$ . Hence  $(1, s) \in \rho(G, S_1)$  and  $(1, s) \notin \rho(G, S_2)$ , i.e.  $\varphi(S_1) \neq \varphi(S_2)$  and  $\varphi$  is injective.

**Corollary (3.5)**

From Theorem (3.4) it results for  $S_i \in \mathcal{S}(G)$ ,  $i \in I$ , that  $\bigcap_{i \in I} S_i$ ,  $\bigcup_{i \in I} S_i \in \mathcal{S}(G)$  and hold

$$(G, \rho(G, \bigcap_{i \in I} S_i)) = \bigcap_{i \in I} (G, \rho(G, S_i)) \quad (3.7)$$

$$(G, \rho(G, \bigcup_{i \in I} S_i)) = \bigcup_{i \in I} (G, \rho(G, S_i)) \quad (3.8)$$

**Proof.** :

Because the lattice  $\mathcal{S}(G)$  is complete and  $\varphi$  is an isomorphism, we have:

$$(G, \rho(G, \bigcap_{i \in I} S_i)) = \varphi(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \varphi(S_i) = \bigcap_{i \in I} (G, \rho(G, S_i)).$$

Analogously, we can prove (3.8).

**Remark (3.6)**

According to the isomorphism between  $\mathcal{S}(G)$  and  $\mathcal{R}(G)$ , which implies the isomorphism between ordered lattice, for  $S_1, S_2 \in \mathcal{S}(G)$ , we have:

$$1) \text{ If } S_1 \subseteq S_2, \text{ then } (G, \rho(G, S_1)) \subseteq (G, \rho(G, S_2)) \quad (3.9)$$

$$2) S_1 \cap S_2 = \phi \text{ implies } (G, \rho(G, S_1)) \cap (G, \rho(G, S_2)) = (G, \rho\phi) \quad (3.10)$$

**Theorem (3.7)**

The Boole algebras  $\mathcal{S}(G)$  and  $\mathcal{R}(G)$  are atomic.

**Proof.**

In the lattice  $\mathcal{S}(G)$ , the atomic elements are:  $S = \{1\}$ , if  $s = 1$  or  $s^2 = 1$  and  $S = \{s, s^{-1}\}$ , if  $s^2 \neq 1$  (order  $s > 2$ ). We assume that an atomic element  $S$ ,  $|S| > 2$  exists. If  $s \in S$ , then  $s^2 = 1$  or  $s^2 \neq 1$ , hence  $\phi < \{s\} < S$  or  $\phi < \{s, s^{-1}\} < S$ , i.e.,  $S$  is not atomic element.

Using the isomorphism between lattices  $\mathcal{S}(G)$  and  $\mathcal{R}(G)$ , it is obtained that  $(G, \rho(G, \{s\}))$ , if  $s^2 = 1$  and  $(G, \rho(G, \{s, s^{-1}\}))$ , if  $s^2 \neq 1$ , are atomic elements in the lattice  $\mathcal{R}(G)$ .

**Remark (3.8)**

If  $s^2 = 1$ , then  $s = s^{-1}$  and  $\{s, s^{-1}\} = \{s\}$ , hence  $(G, \rho(G, \{s, s^{-1}\}))$  contains  $(G, \rho(G, \{s\}))$ .

**Theorem (3.9)**

Every  $G$ -related system can be uniquely represented as a union of indecomposable  $G$ -related systems.

**Proof.**

It is evident that the atomic elements are indecomposable.

Let  $(G, \rho(G, S)) \in \mathcal{R}(G)$ . In the lattice  $\mathcal{S}(G)$ ,  $S$  is uniquely written as a union of atomics.

$$S = \bigcup_{s \in S} \{s, s^{-1}\} \quad (3.11)$$

According to (3.8), we have

$$(G, \rho(G, S)) = \bigcup_{s \in S} (G, \rho(G, \{s, s^{-1}\})) \quad (3.12)$$

**Theorem (3.10)**

Every  $G$ -related system can be uniquely represented as an intersection of irreducible  $G$ -related systems.

**Proof.**

Let  $(G, \rho(G, T)) \in \mathcal{R}(G)$ . If we denote  $S = G \setminus T$ , then  $T = S'$ , hence, according to (3.3) and (3.8) we have:

$$(G, \rho(G, T)) = (G, \rho(G, S')) = \bigcup_{s \in S} [(G, \rho(G, \{s, s^{-1}\}))]'$$

or

$$(G, \rho(G, T)) = \bigcap_{s \in S} (G, \rho(G, \{s, s^{-1}\}))'.$$

## References

- [1] St. Antohe, *Properties of some Graph Categories*, Buletin Univerşity of Galati, fasc.II, pp. 35 - 39, 1986.
- [2] St. Antohe, *Quasi-Cayley Graphs*, Analele Univ. of Galati, fasc. II, pp. 28-32, 1989.
- [3] St. Antohe, *The binary-related system. Category properties*, Analele Univ. of Galati, fasc.II, pp. 41-47, 1993.
- [4] St. Antohe and E. Olaru, *On the Structure of Quasi-Cayley Graphs. Decomposition Theorems*, Libertas Mathematica, vol. XIV, Arlington, Texas, pp. 137-149, 1994.
- [5] St. Antohe, *The Circulant Graphs. Applications*, Libertas Mathematica, vol. XVIII, Arlington, Texas, pp.165-174, 1997.
- [6] C. Nastasescu, *Dimension Theory in Noncommutative Algebra*, Ed. Acad. R. S. Romania, Bucuresti, 1983.

