LIBERTAS MATHEMATICA, VOL XX (2000)

SOME NEW CLASSES OF CONNECTED MATROIDS

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Abstract

The connected matroid concept was introduced since 1935, by Whitney, in [21], and it was extended by Tutte, in [15], starting mostly from 3-connected graphs, which he studied in [14]. This paper, uses the simplest and the customariest definition for the connected matroids, as in [18]. With this, and going away from n-connected graphs, $n \geq 2$, or from p-regular graphs, $p \in \mathbf{N}^*$, we shall find some new connected matroid classes, for the cycle matroid of a graph.

AMS classification: 05B35, 05C40

Key words: articulation-vertex, circuit, cocycle, connected, connectivity, cycle, graph, matroid, rank, regular.

In the following, we shall use unoriented graphs $G = (V(G), E(G))$ and their corresponding cycle matroids of G, denoted by $M(G) = (S = E(G), F)$, as in [18] and [19].

Definition 1. i) *The graph G, with at least $n+1$ vertices, is n-connected, if $n \in N^*$ is the minimal number of vertices, so that removing them and all their incident edges, G becomes disconnected or trivial.*

ii) *G is p-regular, $p \in N^*$, if any $v \in V(G)$ has degree $d(v) = p$.* ■

Remark 1. The n-connected graph concept was firstly introduced by Whitney, in [20]. We shall meet it again, through a similar form, in [1], [6], [7], [12], [13], [18]. In [17], the n-connected graph definition is equivalently given, by the notion of n-separability. Usually, the n-connected graph G is named vertex-n-connected (see [6], [16], [17]). We must notice, that Whitney defined in [20], the connectivity $k(G)$, of a graph G, as a property, when G is n-connected but is not $(n+1)$-connected, being the definition, currently used in the Graph Theory. ■

From [17] and [18] we know:

Definition 2. *The matroid $M = (S, F)$ is connected, if for any $x \neq y \in S$, a circuit exists in M, containing x and y* ■

Remark 2. i) Whitney introduced the anterior concept, since 1935. In accordance with [21], a matroid $M = (S, F)$ if for any $A \neq \emptyset$, $A \subset S$, we have:

$$r(A) + r(S \backslash A) > r(S) \tag{1}$$

where r represents the rank-function on M.

ii) In [18], Welsh showed that the matroid $M = (S, F)$ is not connected, iff there is $A \neq \emptyset$, $A \subset S$, so that:

(2) $$r(A) + r(S \backslash A) = r(S)$$

iii) From [18] and [19], we know that in the cycle matroid $M(G)$, of the graph G, its circuits are cycles of G.

As in [18], we introduce:

Definition 3. *The graf G is a block, if its cycle matroid $M(G)$ is connected.* ■

From [18], with a complete proof in [9], we have:

Theorem 1. *Let G be a connected graph, without loops and at least 3 vertices. Then, the following assertions are equivalent:*

i) *G is a block;*

ii) *G is 2-connected;*

iii) *every 2 vertices of G belong to a common cycle;*

iv) *every vertex and every edge of G belong to a common cycle.* ■

A natural consequence of this theorem is:

Corollary 1. *The graph G is 2-connected, iff $M(G)$, the cycle matroid of G, is connected.* ■

Remark 3. **i**) The **Definition 2** and the **Corollary 1** permit us to state as in [4], that:

"a graph G is a block, iff G is 2-connected".

ii) Surely, if G is a block, it has not articulation vertices or cut-vertices, namely vertices, whose removal increases the number of the connected components. ■

In [11], an extension of a **Theorem's 1** part was presented, having as starting point, the idea that a n-connected graph is not automaticaly $(n-1)$-connected. This idea was mentioned, since 1966, by Tutte's exemple in which he constructed a 2-connected graph, which is not 3-connected, using the generalized cycle notion (see [16]). Tutte's example is also quoted in [12]. We shall take again our extension, giving another proof of its first assertion.

Theorem 2. *For the natural number $n \geq 3$, let G be a n-connected graph, with at least 4 vertices and without loops. Then, the following statements are true:*

i) *every 2 vertices of G belong to a common cycle;*

ii) *every vertex and every edge of G belong to a common cycle.*

Proof. **i**) By reductio ad absurdum we suppose that the vertices $v_1 \neq v_2 \in V(G)$ are out of any cycle C of G. We can place us in one of the 3 situations presented in the **Figure1**, all other possibilities being reductible to these situations.

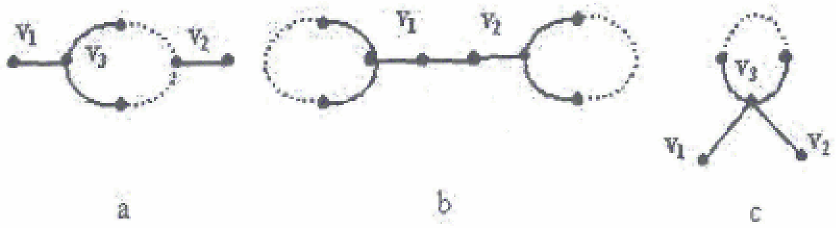

Figure 1

In the cases a and c, by eliminating the vertex v_3, we get an 1-connected graph.

Also, for the case b, by eliminating the vertex v_1 or v_2, a similar 1-connected graph is obtained. But together, these cases would be in contradiction with the hypothesis, because G is n-connected, with $n \geq 3$. Consequently, our assumption is false, and we decide that any $v_1 \neq v_2 \in V(G)$ belong to a common cycle.

ii) By reductio ad absurdum we suppose that $v \in V(G)$ and $x \in E(G)$ do not belong to any cycle C, of G. There are 3 situations, as in the **Figure 2**, each of them depending on the adjacency of v with x.

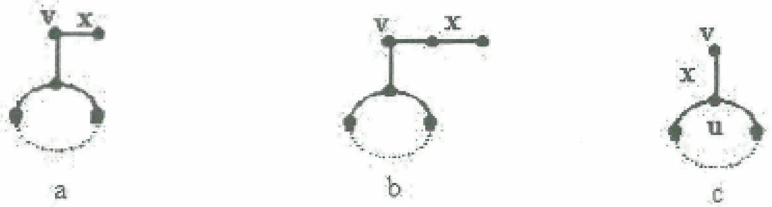

Figure 2

In the cases a and b, we immediately state that removing v increases the number of the connected components of G. A similar situation appears in the case c, but, for the vertex u. Therefore, G would be 1-connected, in contradiction with the hypothesis! We deduce that our assumption is false, and so, any vertex v and any edge x belong to a common cycle. ■

Remark 4. For other extensions of the **Theorem 1**, different from the **Theorem 2**, and relating to the n-connection with $n \geq 3$, see [2], [5], [12].

By means of the **Theorem 2**, we can find a new connected matroid class.

Theorem 3. *For the natural number $n \geq 3$, let G be a n-connected graph with at least 4 vertices and without loops. Then $M(G)$, the cycle matroid of G, is connected.*

Proof. By reductio ad absurdum we suppose that $M(G)$ is not connected. So, we have $x \neq y \in S$,, which do not belong to any circuit of $M(G)$, hence, to any cycle of G. It is possible that any of the 3 situations presented in the **Figure 3** hold, all the others, being reductible to these cases.

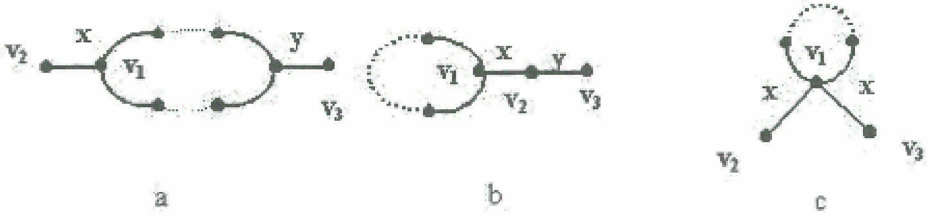

Figure 3

We immediately notice, that deleting the vertex v_1 (which is an articulation), in each of the 3 cases of the **Figure 3**, we are led to a graph in which the number of the connected components increases.So we find that G is 1-connected,in contradiction to the hypothesis! Now, we can state that the initial assumption is false, and consequently, any $x \neq y \in S$ belong to the same circuit of $M(G)$. ■

Remark 5. i) In [11] we established the proof of the **Theorem 3**, through the **Theorem 2**. So, all the 3 situations from the **Figure 3**, show us that, the vertices v_2 and v_3, or also, the edge x and the vertex v_2 are not in a common cycle of G, these facts contradicting the assertions of the **Theorem 2**.

ii) The reciprocal theorem for the **Theorem 3** is false ! Indeed, the graph G, from the **Figure 4**, is only 2-connected, while $M(G)$, the cycle matroid of G, is connected.

iii) The **Theorem 3** comes to see us that, through $M(G)$, the cycle graph matroid, the *n-connectivity* of G implies the $n-1$ *connectivity*, fact which is not generally true, how Tutte showed in [16]! ■

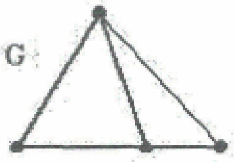

Figure 4

Figure 5

Going on, we shall state, that if a graph G is connected (or other saying, 1-connected), this does not compulsory mean the connectivity for $M(G)$, the cycle matroid of G. A simple example is given in the **Figure 5**.

In the following, we want to study the connection between p-regularity, connectivity and n-connectivity of a graph G, respectively the connectivity for a graph G, respectively the connectivity for $M(G)$, the cycle matroid of G, on condition that from now on, we shall work with simple graphs.

From [3] (see a new proof in [9]), we know:

Theorem 4. *For $p \in N^*$, let G be a simple and p-regular graph. Then the following assertions are hold*:

i) *p is an eigenvalue of G;*

ii) *if in addition, G is connected, the multiplicity of p is 1;*

iii) *for any other eigenvalue $\lambda \neq p$, of G, the inequality $|\lambda| \leq p$ occurs.* ■

In [9] and [10] we showed that the **Theorem 4** accepts a reciprocal theorem, only for the statement **ii)**, and this appears in the next:

Theorem 5. *For $p \in \mathbf{N}^*$ let G be a simple and p-regular graph, which has the eigenvalue p, with the multiplicity 1. Then, G is connected.* ■

As we saw above, because the connectivity of G does not imply the connectivity for $M(G)$, the cycle matroid of G, we shall try to add some new conditions to the p-regularity, so that $M(G)$ becomes connected.

Let us firstly notice that the p-regularity, of a connected multigraph G, does not imply the connectivity of the matroid $M(G)$. A good example for this is the 3-regular graph presented in the **Figure 6**. We shall use the relation (2), proving that $M(G)$ is not

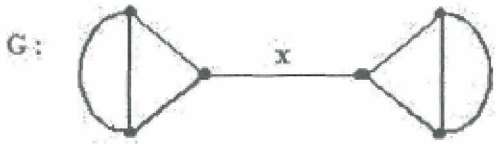

Figure 6

connected. Let take $A = \{x\}$. Because in $M(G)$, the set A is independent, it follows that $r(A) = 1$. The rank of the matroid $M(G)$ is $r(M(G)) = r(S) = 5$, and the rank of the set $S \backslash A$ is $r(S \backslash A) = 4$, because in these two connected components, which result by removing the edge x, we can find independent sets, with two elements. Therefore, the relation (2) is proved, and the multigraph from the **Figure 6**, which is 3-regular and 1-connected, does not generate a connected matroid $M(G)$.

There are also simple, p-regular and n-connected graphs, so that $M(G)$, the cycle matroid of G, is not connected. A such example, in which G is simple, 3-regular and 1-connected, is presented in the **Figure 7**.

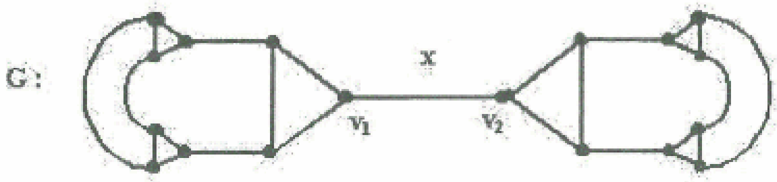

Figure 7

The matroid $M(G)$ is not connected. Indeed, accordingly to a theorem from [18], there is no circuit which contains $x \in S = E(G)$, another proof being given as in the case of the graph drawn in the **Figure 6**.

Remark 6. **i)** From [2] or [4], we know that v_1 and v_2, incident with the edge x, of the **Figure 7** are articulation vertices. For these vertices and for the connectivity $k(G)$ of a graph G, the next property holds:

"G is connected and it has no articulation vertex, iff the connectivity $k(G) \geq 2$",

also admiting the equivalent form:

"G is connected and it has articulation vertices, iff $k(G) \leq 1$".

In addition, we state that the graph G, of the **Figure 7**, agrees with this last property !

ii) The same **Figure 7** comes to invalidate the **Proposition 4**, from [10]. ■

The **Corollary 1**, the **Theorem 3** and the **Remark 6 i)** permit us to establish:

Theorem 6. *Let G be a connected graph with at least 4 vertices and without loops. If the connectivity $k(G) \geq 2$, then $M(G)$, the cycle matroid of G, is connected.* ■

Surely, the **Theorem 5** and the **Theorem 6** give us:

Corollary 2. *For $p \in \mathbf{N}^*$, let G be a simple and p-regular graph having at least 4 vertices, and admiting the eigenvalue p, with the multiplicity 1. If in addition, the connectivity $k(G) \geq 2$, then $M(G)$, the cycle matroid of G, is connected.* ■

Another new connected matroid class will be presented below:

Theorem 7. *Let G be a simple and p-regular graph, with $0 < p < n$ and having $4 \leq |V(G)| = n$. If $p \in \{n-2, n-1\}$, then $M(G)$, the cycle matroid of G, is connected.*

Proof. From [2], we deduce that if, for any $0 < p < n$ and for any $v \in V(G)$, we have:

$$d(v) \geq \frac{n+p}{2} - 1, \tag{3}$$

then, G is p-connected. Because G is p-regular, the relation (3) becomes:

$$p \geq \frac{n+p}{2} - 1, \text{ or equivalently, } p \geq n-2 \tag{4}$$

In addition with $p < n$, we obviously have $p \in \{n-2, n-1\}$ and the property quoted above allows us to state that G is p-connected.

On the other hand, because $n \geq 4$, the p-connectivity of G means that G is only 2-connected, or 3-connected. From here, with the **Corollary 1** and the **Theorem 3**, we obviously have that $M(G)$ is connected. ■

Going on, we shall study if there are also other forms equivalent to the **Corollary 2**. For this, we shall need to see the connection, between the cocycles of G and its connectivity $k(G)$.

Accordingly with [6], [8], [12], [13], [18], we introduce:

Definition 4. *The cocycle, of a graph G, is a minimal edge-set $A \subset E(G)$, whose removal from G, increases the number of its connected components.* ■

Remark 7. In [12], Oxley uses bond and Recski, in [13], prefers the notion of minimal cut-set. At Berge, the cocycle is defined in [12] through an equivalent manner. ■

In the following, we shall consider the study of the connected graph G, from the **Figure 8**.

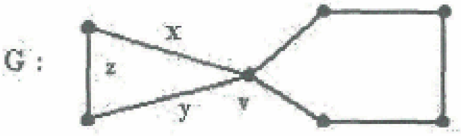

Figure 8

G has $\{x, y\}$, as cocycle with the length 2, but it is not 2-connected, because v is articulation vertex. Therefore, we conclude that the existence of a cocycle with the length 2, in a graph G, does not imply the 2-connectivity for G ! The matroid $M(G)$ is also not connected! Indeed, using the relation (2), there is $A = \{x, y, z\}$, with the rank $r(A) = 2$. Because $r(S \backslash A) = 4$ and $r(M(G)) = r(S) = 6$, the matroid $M(G)$ is not connected, the relation (2) being true.

This last statement allows us to establish:

Theorem 8. *Let G be a simple and connected graph, with at least 3 vertices and having only cocyles whose length is 2. Then $M(G)$, the cycle matroid of G, is not compulsorily connected.*

Proof. G being simple, only with cocycles of the length 2, we have 3 posibilities.

i) G is a block, case in which G is 2-connected, hence $M(G)$ is connected.

ii) G has an articulation vertex as the graph of the **Figure 8**. With similar arguments as above, we decide that $M(G)$ is not connected.

iii) G is a reunion between a block, and an other part, having an articulation vertex. $M(G)$ is not connected, using arguments as in **ii**). ■

From the **Theorem 5** and **Theorem 8**, we deduce:

Corollary 3. *For $p \in \mathbf{N}^*$, let G be a simple and p-regular graph, having at least 3 vertices, and the eigenvalue p, with the multiplicity 1. If G admits only cocycles, whose length is 2, then $M(G)$, the cycle matroid of G, is not compulsorily connected.* ■

References

[1] **Behzad M., Chartrand G., Lesniack-Foster L.**, "*Graphs and digraphs*", Weber & Schmidt, International Series, 1979;

[2] **Berge C.**, "*Graphs*", North Holland, 1989;

[3] **Biggs N.**, "*Algebraic graph theory*", Cambridge University Press, 1974;

[4] **Brualdi R. A.** "*Introductory combinatorics*", North Holland, 1977;

[5] **Capobianco M., Moluzzo J. C.** "*Examples and counterexamples in graph theory*",North Holland, 1978;

[6] **Deo N.**, "*Graph theory with applications to engineering and computer science*", Prentice Hall Inc., 1974;

[7] **Frank A.** "*Connectivity and network flows" in "Handbook of Combinatorics*", ed. by Graham R., Grötschell M.,Lovàsz L., Elsevier Sc. B. V., 1995 pg. 111-77;

[8] **Harary F., Welsh D. J. A.**, "*Matroids versus graphs*", in Lectures Notes in Mathematics, 110, "The many facets of graph theory", Springer-Verlag, 1969, pg. 155-70;

[9] **Modan L.**, "*Les réciproques d'un théoréme sur les graphes connexes et p-réguliers*", An. Univ. Timisoara, v.31, fasc.1, 1993, pg.71-6;

[10] **Modan L.**, "*Asupra unor proprietati ale grafurilor conexe p-regulate*", (in Romanian), St. si Cerc. de Calc. Ec. si Cib. Ec., v.30, nr. 4, 1996, pg. 39-43;

[11] **Modan L.**, "*N-connected graphs and connected matroids*", Ec.Comp. and Ec. Cyb., St. and Res., v.30, nr. 1-4,1996, pg. 91-4;

[12] **Oxley J.**, "*Graphs and series-paralel network*" in "Theory of matroids" ed. by White N., Cambridge University Press, 1986, pg. 97-125;

[13] **Recski A**, "*Matroid theory and its applications in electric network theory and in statics*", Springer Verlag, 1989;

[14] **Tutte W. T.**, "*A theory of 3-connected graphs*", Indag. Maths., 23, 1961, pg.441-55;

[15] **Tutte W. T.**, "*Connectivity in matroids*", Canadian J. of Maths., 18, 1966, pg. 1301-24;

[16] **Tutte W. T.**, "*Connectivity in graphs*", University of Toronto Press, 1966;

[17] **Tutte W. T.**, "*Graph Theory*", Adison Wesley Publishing Company,1984;

[18] **Welsh D.J.A.**, "*Matroid theory*", Academic Press, 1976;

[19] **Welsh D.J.A.**, "*Matroids. Fundamental Concepts*" in "Handbook of Combinatorics", ed. by Graham R., Grötschell M.,Lovàsz L., Elsevier Sc. B.V., 1995, pg. 483-523;

[20] **Whitney H.**, "*Congruent graphs and the connectivity of graphs*", American J. Maths., nr. 54, 1932, pg. 151-68;

[21] **Whitney H.**, "*On the abstract properties of linear dependence*", American J. Maths., nr. 57, 1935, pg. 509-33