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DYNAMIC FLOWS WITH SUPPLY AND DEMAND IN NETWORKS WITH SEVERAL SOURCE AND SINK NODES

ELEONOR CIUREA

Abstract

Given a network $G = (N, A, h, c)$ with node set N, arc set A, time function h, capacity function c and P the set of periods, partition the nod set N into the disjoint source set I, intermediate nod set J and the sink set K. Associate with each $x \in I$ and each $t \in P$ a nonnegative real number $u(x;t)$ called the supply, and with each $x \in K$ and each $t \in P$ a nonnegative real number $w(x;t)$ called the demand. The objective is to determine the existence of a dynamic flow in G, so that the demands at the sink set K can be fulfilled from the supplies at the source set I. A numerical example is presented.

1. Dynamic flows in networks with several source and sink nodes

Let $G = (N, A)$ be a conected digraf with N the node set and A the arc set. Let $\mathcal{N}$ be the set of natural numbers and $P = \{0, 1, ..., p\}$ be the set of periods. Let us state the time function $h : A \to \mathcal{N}$ and the capacity function $c : A \times P \to \mathcal{N}$ where $h(x,y)$ represents the arc transit time and $c(x,y;t)$ the arc capacity at time $t \in P$ for $(x,y) \in A$. We partition the node set N into the disjoint source set I, intermediate node set J, and the sink set K.

The dynamic flows problem from I to K for p time periods may be formulated as follows. Let us determine the function $f : A \times P \to \mathcal{N}$, which should satisfy the following relations:

$$\sum_{t=0}^{p}\left(\sum_{y} f(x,y;t) - \sum_{y} f(y,x;t')\right) = v(x;P),\ x \in I \tag{1}$$

$$\sum_{y} f(x,y;t) - \sum_{y} f(y,x;t') = 0,\ x \in J,\ t, t' \in P \tag{2}$$

$$\sum_{t=0}^{p}\left(\sum_{y} f(x,y;t) - \sum_{y} f(y,x;t')\right) = -v(x;P),\ x \in K \tag{3}$$

$$0 \leq f(x,y;t) \leq c(x,y;t),\ (x,y) \in A,\ t \in P, \tag{4}$$

$$v(P) = \sum_I v(x; P) = \sum_K v(x; P) \tag{5}$$

where $t' = t - h(y, x)$.

In the reference [4] is shown that a dynamic flow for p time periods in the static network $G = (N, A, h, c)$, $N = I \cup J \cup K$, is equivalent with a dynamic flow for p time periods in the static network $\tilde{G} = (\tilde{N}, \tilde{A}, \tilde{h}, \tilde{c})$, where:

$\tilde{N} = N \cup \hat{N}, \hat{N} = \{\hat{i}, \hat{k}\}$;

$\tilde{A} = A \cup \hat{A}, \hat{A} = \{(\hat{i}, x) | x \in I\} \cup \{(x, \hat{k}) | x \in K\}$;

$\tilde{h} : \tilde{A} \to \mathcal{N},\ \tilde{h}(x, y) = h(x, y),\ (x, y) \in A,\ \tilde{h}(x, y) = 0,\ (x, y) \in \hat{A}$;

$\tilde{c} : \tilde{A} \to \mathcal{N},\ \tilde{c}(x, y; t) = c(x, y; t),\ (x, y) \in A,\ t \in P$,

$\tilde{c}(x, y; t) = \infty,\ (x, y) \in \hat{A},\ t \in P$.

We call node $\hat{i}$ the supersource and node $\hat{k}$ the supersink. Ford and Fulkerson, [5], have shown that a dynamic flow for p time periods in the static network $\tilde{G} = (\tilde{N}, \tilde{A}, \tilde{h}, \tilde{c})$ can be represented as a static flow in the dynamic network $\tilde{G}(p) = (\tilde{N}(p), \tilde{A}(p), \tilde{c})$, where:

$$\tilde{N}(p) = \{x(t) | x \in \tilde{N}, t \in P\},$$
$$\tilde{A}(p) = \{(x(t), y(t')) | (x, y) \in \tilde{A}; t, t' \in P, t' = t + h(x, y)\},$$
$$\tilde{c}(x(t), y(t')) = c(x, y; t), (x, y) \in \tilde{A}; t, t' \in P.$$

In the references [2], [3] it is shown that a dynamic flow for p time periods in the static network $\tilde{G} = (\tilde{N}, \tilde{A}, \tilde{h}, \tilde{c})$ is equivalent with a dynamic flow for the same time periods in the static network $\tilde{G}'(p) = (\tilde{N}'(p), \tilde{A}'(p), \tilde{h}', \tilde{c}')$ and can be represented as a static flow in the dynamic network:

$$\tilde{G}^*(p) = (\tilde{N}^*(p), \tilde{A}^*(p), \tilde{c}^*).$$

The network $\tilde{G}'(p)$ and $\tilde{G}^*(p)$ may be constructed as follows. Let $\tilde{d}(\hat{i}, x)$ be the length of the shortest route from the supersource $\hat{i}$ to the node x and $\tilde{d}(x, \hat{k})$ the length of a shortest route from node x to the supersink $\hat{k}$, with respect to the function $\tilde{h}$. Let us consider:

$$\tilde{P}(x) = \{t | t \in P, \tilde{d}(\hat{i}, x) \leq t \leq p - \tilde{d}(x, \hat{k})\}, x \in \tilde{N},$$
$$\tilde{P}(x, y) = \{t | t \in P, \tilde{d}(\hat{i}, x) \leq t \leq p - (\tilde{h}(x, y) + \tilde{d}(x, \hat{k}))\}, (x, y) \in \tilde{A}.$$

The network $\tilde{G}'(p) = (\tilde{N}'(p), \tilde{A}'(p), \tilde{h}', \tilde{c}')$ may be constructed as follows

$\tilde{N}'(p) = \{x | x \in \tilde{N}, \tilde{P}(x) \neq \emptyset\}$,

$\tilde{A}'(p) = \{(x, y) | (x, y) \in \tilde{A}, \tilde{P}(x, y) \neq \emptyset\}$, and $\tilde{h}', \tilde{c}'$ are the restrictions of $\tilde{h}$ and $\tilde{c}$, respectively, to $\tilde{A}'(p)$.

The network $\tilde{G}^*(p) = (\tilde{N}^*(p), \tilde{A}^*(p), \tilde{c}^*)$ is constructed from network $\tilde{G}'(p)$ in the following manner:

$\tilde{N}^*(p) = \{x(t) | x \in \tilde{N}'(p), t \in \tilde{P}(x)\}$,

$\tilde{A}^*(p) = \{(x(t), y(t')) | (x, y) \in \tilde{A}'(p), t \in \tilde{P}(x, y), t' = t + h(x, y)\}$,

$$\tilde{c}^*(x(t), y(t)) = c(x, y; t), (x, y) \in \tilde{A}'(p), t \in \tilde{P}(x, y).$$

The network $\tilde{G}'(p)$ is, generaly, a partial subnetwork of $\tilde{G}$ and $\tilde{G}^*$ is always a partial subnetwork of $\tilde{G}(p)$.

A dynamic flow problem is said to be stationary if the network parameters such as capacities, arc traversal times, and so on, are constant over time ($h : A \to \mathcal{N}, c : A \to \mathcal{N}$, and so on).

In the references [2], [3], [5] are presented the algoritms for dynamic flow problem. In the stationary case one does not require the construction of the time-expanded network $\tilde{G}(p)$ or $\tilde{G}^*(p)$ for solving this problem for any p. In these references it is shown that a dynamic flow in the stationary case can be generated from a static flow in network $\tilde{G}$ or $\tilde{G}'(p)$.

A dynamic flow for p time periods in the static network $G = (N, A, h, c)$, $N = I \cup J \cup K$ is equivalent with a dynamic flow for the same time periods in the static network $G'(p) = (N'(p), A'(p), h', c')$ and can be represented as a static flow in the dynamic network $G^*(p) = (N^*(p), A^*(p), c^*)$, where:

$$N'(p) = \tilde{N}'(p) - \hat{N}'(p), \hat{N}'(p) = \{\hat{i}, \hat{k}\};$$

$$A'(p) = \tilde{A}'(p) - \hat{A}'(p), \hat{A}'(p) = \{(\hat{i}, x) | x \in I\} \cup \{(x, \hat{k}) | x \in K\};$$

$$N^*(p) = \tilde{N}^*(p) - \hat{N}^*(p), \hat{N}^*(p) = \{\hat{i}(t) | t \in \tilde{P}(\hat{i})\} \cup \{\hat{k}(t) | t \in \tilde{P}(\hat{k})\};$$

$$A^*(p) = \tilde{A}^*(p) - \hat{A}^*(p), \tilde{A}^*(p) = \{(x(t), y(t)) | (x, y) \in \hat{A}'(p), t \in \tilde{P}(x, y)\};$$

$$P(x) = \tilde{P}(x), x \in N, P(x, y) = \tilde{P}(x, y), (x, y) \in A.$$

2. Dynamic flows with supply and demand in networks with several source and sink nodes

Associate with each $x \in I$ and each $t \in P$ a nonnegative real number $u(x; t)$ called the supply of some commodity at node x and time t, and with each $x \in K$ and each $t \in P$ a nonnegative real number $w(x; t)$ called the demand of some commodity at node x and time t. Thus, the supplies in G can be considered as a function $u : I \times P \to \mathcal{R}_+$ and the demands as a functions $w : K \times P \to \mathcal{R}_+$.

The objective is to determine the existence of a dynamic flow in G, so that the demands at the sink set K can be fulfilled from the supplies at the source set I satisfying the constraints:

$$\sum_y f(x, y; t) - \sum_y f(y, x; t') \le u(x; t), \ x \in I, \ t \in P, \tag{6}$$

$$\sum_y f(x, y; t) - \sum_y f(y, x; t') = 0, \ x \in J, \ t \in P, \tag{7}$$

$$\sum_y f(y,x;t') - \sum_y f(x,y;t) \geq w(x;t),\ x \in K,\ t \in P, \tag{8}$$

$$0 \leq f(x,y;t) \leq c(x,y;t) = 0,\ (x,y) \in A,\ t \in P, \tag{9}$$

where $t' = t - h(y,x)$.

If such a solution exists, we say that the constraints (6)-(9) are feasible. Otherwise, they are infleasible.

In network $G^*(p)$ let us consider:
$X^*(p) \subset N^*(p), \bar{X}^*(p) = N^*(p) - X^*(p), I^*(p) = \{x(t)|x \in I'(p), t \in P(x)\}$,
$J^*(p) = \{x(t)|x \in J'(p), t \in P(x)\},\ K^*(p) = \{x(t)|x \in K'(p), t \in P(x)\}$,
$R^*(p) = I^*(p) \cap \bar{X}^*(p),\ V^*(p) = J^*(p) \cap \bar{X}^*(p),\ S^*(p) = K^*(p) \cap \bar{X}^*(p)$,
$T^*(p) = \{(x(t), y(t'))|(x(t), y(t')) \in A^*(p), x(t) \in X^*(p), y(t') \in \bar{X}^*(p)\}$,
$u^*(x(t)) = u(x;t), x(t) \in I^*(p), w^*(x(t)) = w(x;t), x(t) \in K^*(p)$.

The supply-demand theorem in network $G^*(p)$ is stated as follows

Theorem 1. *The constrains (6)-(9) are fleasible if and only if*

$$\sum_{S^*(p)} w^*(x(t)) - \sum_{R^*(p)} u^*(x(t)) \leq \sum_{T^*(p)} c^*(x(t), y(t')) \tag{10}$$

holds for every subset $X^*(p) \subset N^*(p)$.

Proof. Necessity Assume that there is a dynamic flow in G satisfying the constrains (6)-(9). Rewriting (6)-(9) in $G^*(p)$ as

$$\sum_{y(t')} f^*(x(t), y(t')) - \sum_{y(t')} f^*(y(t'), x(t)) \leq u^*(x(t)), x(t) \in I^*(p), \tag{11}$$

$$\sum_{y(t')} f^*(x(t), y(t')) - \sum_{y(t')} f^*(y(t'), x(t)) = 0, \qquad x(t) \in J^*(p), \tag{12}$$

$$\sum_{y(t')} f^*(y(t'), x(t)) - \sum_{y(t')} f^*(x(t), y(t')) \geq w^*(x(t)), x(t) \in K^*(p), \tag{13}$$

$$0 \leq f^*(x(t), y(t')) \leq c^*(x(t), y(t')), (x(t), y(t')) \in A^*(p), \tag{14}$$

and by summing constrains (11)-(13) over $x(t) \in \bar{X}^*(p)$, yields

$$\sum_{R^*(p)} \sum_{y(t')} f^*(x(t), y(t')) - \sum_{R^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) \leq \sum_{R^*(p)} u^*(x(t)), \tag{15}$$

$$\sum_{V^*(p)} \sum_{y(t')} f^*(x(t), y(t')) - \sum_{V^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) = 0, \tag{16}$$

$$\sum_{S^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) - \sum_{S^*(p)} \sum_{y(t')} f^*(x(t), y(t')) \geq \sum_{S^*(p)} w^*(x(t)). \tag{17}$$

Rewriting (15) and (16) as

$$\begin{aligned} \sum_{R^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) - \sum_{R^*(p)} \sum_{y(t')} f^*(x(t), y(t')) &\geq - \sum_{R^*(p)} u^*(x(t)), \\ \sum_{V^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) - \sum_{V^*(p)} \sum_{y(t')} f^*(x(t), y(t')) &= 0, \end{aligned}$$

and combining these with (17), we obtain

$$\sum_{S^*(p)} w^*(x(t)) - \sum_{R^*(p)} u^*(x(t)) \leq$$

$$\leq \sum_{\bar{X}^*(p)} \sum_{y(t')} f^*(y(t'), x(t)) - \sum_{\bar{X}^*(p)} \sum_{y(t')} f^*(x(t), y(t')). \quad (18)$$

Using $N^*(p) = X^*(p) \cup \bar{X}^*(p)$ and $y(t') \in N^*(p)$ in (18) gives

$$\sum_{S^*(p)} w^*(x(t)) - \sum_{R^*(p)} u^*(x(t)) \leq$$

$$\leq \sum_{\bar{X}^*(p)} \sum_{X^*(p)} f^*(y(t'), x(t)) - \sum_{\bar{X}^*(p)} \sum_{X^*(p)} f^*(x(t), y(t')). \quad (19)$$

with $x(t) \in \bar{X}^*(p)$ and $y(t') \in X^*(p)$, or equivalently

$$\sum_{S^*(p)} w^*(x(t)) - \sum_{R^*(p)} u^*(x(t)) \leq$$

$$\leq \sum_{X^*(p)} \sum_{\bar{X}^*(p)} f^*(x(t), y(t')) - \sum_{X^*(p)} \sum_{\bar{X}^*(p)} f^*(y(t'), x(t)). \quad (20)$$

with $x(t) \in X^*(p)$ and $y(t') \in \bar{X}^*(p)$.

Since f^* satisfies (14), it results

$$\sum_{X^*(p)} \sum_{\bar{X}^*(p)} f^*(x(t), y(t')) - \sum_{X^*(p)} \sum_{\bar{X}^*(p)} f^*(y(t'), x(t)) \leq \sum_{T^*(p)} c^*(x(t), y(t')).$$

Combining this with (20) shows that the constraint (10) must hold for every subset $X^*(p) \subset N^*(p)$.

Sufficiency. To prove sufficiency, we construct a new network
$\tilde{G}_1 = (\tilde{N}_1, \tilde{A}_1, \tilde{h}_1, \tilde{c}_1)$ from $\tilde{G} = (\tilde{N}, \tilde{A}, \tilde{h}, \tilde{c})$, where
$\tilde{N}_1 = \tilde{N}, \tilde{A}_1 = \tilde{A}$,
$\tilde{h}_1 = \tilde{h}, \tilde{c}_1(x, y; t) = c(x, y; t), (x, y) \in A, t \in P, \tilde{c}_1(\hat{i}, x; t) = u(x; t), x \in I$,
$\tilde{c}_1(x, \hat{k}; t) = w(x; t), x \in K, t \in P$.

We now show that the inequality (10) holds for every subset
$X^*(p) \subset N^*(p)$ if and only if it holds for the cut
$\tilde{T}_1^*(\hat{k}) = \{(x(t), \hat{k}(t'))/x \in K, t \in \tilde{P}_1(x), t' \in \tilde{P}_1(\hat{k})\}$ in
$\tilde{G}_1^*(p) = (\tilde{N}_1^*(p), \tilde{A}_1^*(p), \tilde{c}_1^*)$.

The implication is that $\tilde{T}_1^*(\hat{k})$ is a minimum $\hat{i}(t) - \hat{k}(t')$ cut in $\tilde{G}_1^*(p)$. To see this, let

$$\tilde{T}_1^*(p) = \{(x(t), y(t'))/(x(t), y(t')) \in \tilde{A}_1^*(p), x(t) \in \tilde{X}_1^*(p), y(t') \in \bar{\tilde{X}}_1^*(p)\}$$

be any $\hat{i}(t) - \hat{k}(t')$ cut in $\tilde{G}_1^*(p)$, and define
$X^*(p) = \tilde{X}_1^*(p) - \{\hat{i}(t)/t \in \tilde{P}_1(\hat{i})\}, \bar{X}^*(p) = \bar{\tilde{X}}_1^*(p) - \{\hat{k}(t')/t' \in \tilde{P}_1(\hat{k})\}$,
$T^*(p) = \{(x(t), y(t'))/(x(t), y(t')) \in A^*(p), x(t) \in X^*(p), y(t') \in \bar{X}^*(p)\}$,
$\tilde{T}^*(\hat{i}) = \{(\hat{i}(t), x(t'))/t \in \tilde{P}_1(\hat{i}), x(t') \in X^*(p)\}$
$\tilde{T}^*(\hat{k}) = \{(x(t), \hat{k}(t'))/t' \in \tilde{P}_1(\hat{k}), x(t) \in \bar{X}^*(p)\}$.

Consider the expanssion

$$\sum_{\tilde{T}_1^*(p)} \tilde{c}_1^*(\hat{i}(t), x(t')) - \sum_{\tilde{T}_1^*(\hat{k})} \tilde{c}_1^*(x(t), \hat{k}(t')) =$$

$$= \sum_{\tilde{T}^*(\hat{i})} \tilde{c}_1^*(\hat{i}(t), x(t')) + \sum_{T^*(p)} \tilde{c}_1^*(x(t), y(t')) +$$
$$+ \sum_{\tilde{T}^*(\hat{k})} \tilde{c}_1^*(x(t), \hat{k}(t')) - \sum_{\tilde{T}_1^*(\hat{k})} \tilde{c}_1^*(x(t), \hat{k}(t')) =$$
$$= \sum_{T^*(p)} c^*(x(t), y(t')) + \sum_{R^*(p)} u^*(x(t)) - \sum_{S^*(p)} w^*(x(t)).$$

By assumption, (10) holds for every subset $X^*(p) \subset N^*(p)$. Thus the inequality

$$(21) \qquad \sum_{\tilde{T}_1^*(p)} \tilde{c}_1^*(\hat{i}(t), x(t')) - \sum_{\tilde{T}_1^*(\hat{k})} \tilde{c}_1^*(x(t), \hat{k}(t')) \geq 0,$$

holds for all $\hat{i}(t) - \hat{k}(t')$ cuts $\tilde{T}_1^*(p)$ of $\tilde{G}_1^*(p)$, showing that $\tilde{T}_1^*(\hat{k})$ is a minimum $\hat{i}(t) - \hat{k}(t')$ cut in $\tilde{G}_1^*(p)$. Our conclusion is that (10) holds for all $X^*(p) \subset N^*(p)$ if and only if (21) holds for all $\hat{i}(t) - \hat{k}(t')$ cuts in $\tilde{G}_1^*(p)$.

Since $\tilde{T}_1^*(\hat{k})$ is a minimum $\hat{i}(t) - \hat{k}(t')$ cut in $\tilde{G}_1^*(p)$, by the max-flow min-cut theorem there exists a flow $\tilde{f}_1^*$ from $\hat{i}(t)$ to $\hat{k}(t')$ in $\tilde{G}_1^*(p)$ that saturates all the arcs of $\tilde{T}_1^*(\hat{k})$. Define

$$f^*(x(t), y(t')) = \tilde{f}_1^*(x(t), y(t')), (x(t), y(t')) \in A^*(p).$$

Clearly, f^* satisfies (12) and (14). To see that f^* satisfies (11) and (13), we consider for all $x(t) \in I^*(p)$ the equations

$$\tilde{f}_1^*(\hat{i}(t'), x(t)) = \sum_{y(t')} \tilde{f}_1^*(x(t), y(t')) - \sum_{y(t')} \tilde{f}_1^*(y(t'), x(t)) =$$
$$= \sum_{y(t')} \tilde{f}^*(x(t), y(t')) - \sum_{y(t')} \tilde{f}^*(y(t'), x(t)),$$

and for all $x(t) \in K^*(p)$ the equations

$$\tilde{f}_1^*(x(t), \hat{k}(t')) = \sum_{y(t')} \tilde{f}_1^*(y(t'), x(t)) - \sum_{y(t')} \tilde{f}_1^*(x(t), y(t')) =$$
$$= \sum_{y(t')} \tilde{f}^*(y(t'), x(t)) - \sum_{y(t')} \tilde{f}^*(x(t), y(t')).$$

Since by construction

$$\tilde{f}_1^*(\hat{i}(t'), x(t)) \leq u^*(x(t)), \tilde{f}_1^*(x(t), \hat{k}(t')) = w^*(x(t),$$

the inequalities (11) and (13) follow. If are satisfies the constraints (11)-(14) then are satisfies the constraints (6)-(9). This completes the proof of the theorem.

In network $G'(p)$ let us consider:
$Q'(x) \subset P(x)$, $\bar{Q}'(x) = P(x) \setminus Q(x)$, $x \in N'(p)$,
$X'(p) = \{x(t)|x \in N'(p), t \in Q'(x)\}, \bar{X}'(p) = \{x(t)|x \in N'(p), t \in \bar{Q}'(x)\}$,
$R'(p) = \{x(t)|x \in I'(p),\ t \in Q'(x)\}$, $S'(p) = \{x(t)|x \in K'(p),\ t \in \bar{Q}'(x)\}$,
$T'(p) = \{(x(t), y(t')|(x, y) \in A'(p),\ x(t) \in X'(x),\ y(t') \in \bar{X}'(p)\}$.

The supply-demand theorem in network $G'(p)$ is stated as follows.

Theorem 2. *The constraints* (6)-(9) *are feasible if and only if*

$$\sum_{S'(p)} w(x;t) - \sum_{R'(p)} u(x;t) \leq \sum_{T'(p)} c'(x(t), y(t')),$$

holds for every subsets $Q'(x) \subset P(x)$, $x \in N'(p)$.

Proof.

This derives directly from Theorem 1 and definition of network $G'(p)$.

If we define the sets P, R, T as $S'(p), R'(p), T'(p)$ then Theorem 2 can be stated in network G as follows

Theorem 3. *The constraints* (6)-(9) *are feasible if and only if*

$$\sum_{S} w(x;t) - \sum_{R} u(x;t) \leq \sum_{T} c(x(t), y(t')),$$

holds for every subsets $Q(x) \subset P(x)$, $x \in N$.

In practice, if we are interested in ascertaining the feasibility of a given supply-demand network $G = (N, A, h, c), N = I \cup J \cup K$ the most efficient way to do this to use the minimum dynamic flow problem in the extended network $\tilde{G}_1 = (\tilde{N}_1, \tilde{A}_1, \tilde{h}_1, \tilde{c}_1)$.

3. Example.

Consider the network $G = (N, A, h, c)$ of Fig.1. The arc transit times are given as the first numbers and the capacities of the arc are given as the second members adjacent to the arcs of Fig.1. The node set N is partitioned

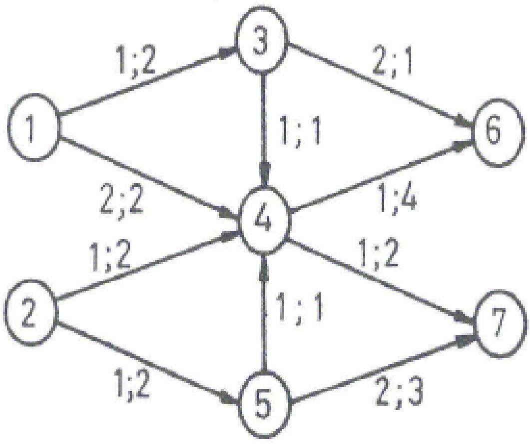

Figure 1

into the source set I, intermediate node set J and the sink set K, as follows:

$$I = \{1, 2\},\ J = \{3, 4, 5\},\ K = \{6, 7\}$$

The set of periods is $P = \{0, 1, 2, 3, 4\}$ and the supplies and demands are given by

$$u(1;0) = 4,\ u(1;1) = 4,\ u(1;2) = 0,\ u(1;3) = 0,\ u(1;4) = 0,$$
$$u(2;0) = 4,\ u(2;1) = 4,\ u(2;2) = 2,\ u(2;3) = 0,\ u(2;4) = 0,$$

$$w(6;0)=0,\ w(6;1)=0,\ w(6;2)=1,\ w(6;3)=4,\ w(6;4)=4,$$
$$w(7;0)=0,\ w(7;1)=0,\ w(7;2)=1,\ w(7;3)=4,\ w(7;4)=4.$$

The network $\tilde{G}_1=(\tilde{N}_1,\tilde{A}_1,\tilde{h}_1,\tilde{c}_1)$ is shown in Fig.2, where on each arc (x,y) is indicated $\tilde{h}_1(x,y)$ and $\tilde{c}_1(x,y;t),\ t\in P$ are

$\tilde{c}_1(0,1;0)=4, \tilde{c}_1(0,1;1)=4, \tilde{c}_1(0,1;2)=0, \tilde{c}_1(0,1;3)=0, \tilde{c}_1(0,1;4)=0;$
$\tilde{c}_1(0,2;0)=4, \tilde{c}_1(0,2;1)=4, \tilde{c}_1(0,2;2)=2, \tilde{c}_1(0,2;3)=0, \tilde{c}_1(0,2;4)=0;$
$\tilde{c}_1(1,3;0)=2, \tilde{c}_1(1,3;1)=2, \tilde{c}_1(1,3;2)=2, \tilde{c}_1(1,3;3)=2, \tilde{c}_1(1,3;4)=2;$
$\tilde{c}_1(1,4;0)=2, \tilde{c}_1(1,4;1)=2, \tilde{c}_1(1,4;2)=2, \tilde{c}_1(1,4;3)=2, \tilde{c}_1(1,4;4)=2;$
$\tilde{c}_1(2,4;0)=2, \tilde{c}_1(2,4;1)=2, \tilde{c}_1(2,4;2)=2, \tilde{c}_1(2,4;3)=2, \tilde{c}_1(2,4;4)=2;$
$\tilde{c}_1(2,5;0)=2, \tilde{c}_1(2,5;1)=2, \tilde{c}_1(2,5;2)=2, \tilde{c}_1(2,5;3)=2, \tilde{c}_1(2,5;4)=2;$
$\tilde{c}_1(3,4;0)=1, \tilde{c}_1(3,4;1)=1, \tilde{c}_1(3,4;2)=1, \tilde{c}_1(3,4;3)=1, \tilde{c}_1(3,4;4)=1;$
$\tilde{c}_1(3,6;0)=1, \tilde{c}_1(3,6;1)=1, \tilde{c}_1(3,6;2)=1, \tilde{c}_1(3,6;3)=1, \tilde{c}_1(3,6;4)=1;$
$\tilde{c}_1(4,6;0)=4, \tilde{c}_1(4,6;1)=4, \tilde{c}_1(4,6;2)=4, \tilde{c}_1(4,6;3)=4, \tilde{c}_1(4,6;4)=4;$
$\tilde{c}_1(4,7;0)=2, \tilde{c}_1(4,7;1)=2, \tilde{c}_1(4,7;2)=2, \tilde{c}_1(4,7;3)=2, \tilde{c}_1(4,7;4)=2;$
$\tilde{c}_1(5,4;0)=1, \tilde{c}_1(5,4;1)=1, \tilde{c}_1(5,4;2)=1, \tilde{c}_1(5,4;3)=1, \tilde{c}_1(5,4;4)=1;$
$\tilde{c}_1(5,7;0)=3, \tilde{c}_1(5,7;1)=3, \tilde{c}_1(5,7;2)=3, \tilde{c}_1(5,7;3)=3, \tilde{c}_1(5,7;4)=3;$
$\tilde{c}_1(6,8;0)=0, \tilde{c}_1(6,8;1)=0, \tilde{c}_1(6,8;2)=1, \tilde{c}_1(6,8;3)=4, \tilde{c}_1(6,8;4)=4;$
$\tilde{c}_1(7,8;0)=0, \tilde{c}_1(7,8;1)=0, \tilde{c}_1(7,8;2)=1, \tilde{c}_1(7,8;3)=4, \tilde{c}_1(7,8;4)=4.$

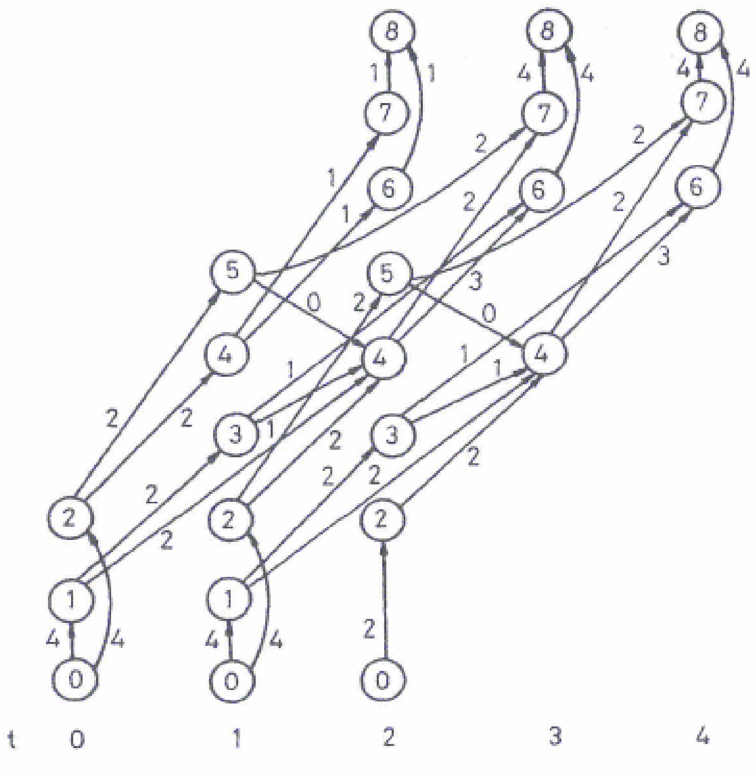

Figure 2

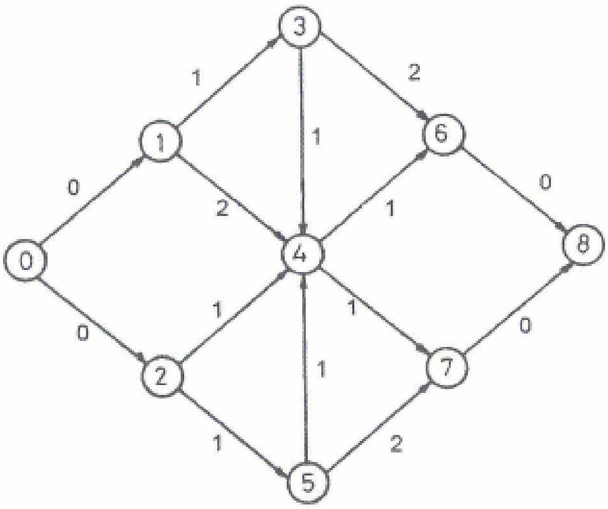

Figure 3

A maxim dynamic flow for $p = 4$ time periods in the static network $\tilde{G}_1$ is represented as a maximum static flow in the dynamic network $\tilde{G}_1^*(4)$ of Fig.3, where the numbers denote their flow value. Since the maximum flow saturates all the arcs of $\tilde{T}_1^*(\tilde{k})$, the problem is therefore feasible.

4. REFERENCES

[1] **AHUJA, R.K.,MAGNATI, T.L., ORLIN, J.B.**, *Network Flows, Theory, Algorithms and Applications*, Prentice Hall, Englewood Cliffs, New Jersey, 1993

[2] **CIUREA, E**, *Deux remarques sur le flot dynamic maximal de cout minimal*, RAIRO, Vert 3, Paris, 1979, 303-306

[3] **CIUREA, E.**, *Les problems des flots dynamiques* Cahiers du CERO, vol.26,no.1-2, Bruxelles, 1984,3-9

[4] **CIUREA, E.**, *Two Extended Concerning the Maximal Dynamic Flows*, ROSYCS, Iasi, 1996, 151-160.

[5] **FULKERSON, D.R., FORD, L.R.**, *Flow in Networks*,Princeton University Press, Princeton, N.Y., 1962

[6] **MURTY, K.G.**, *Network Programing*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.