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TOTAL CURVATURE IN MINKOWSKI PLANES

MOSTAFA GHANDEHARI

Abstract

Generalization of the following result in Euclidean and Minkowski spaces is given. A closed curve of class C^2 in Euclidean n-space which lie in a ball of radius R satisfies the inequality $L \leq RK$ where L is the length of the curveis and K is its total curvature.

1. Introduction

Let C be a closed curve of class C^2 in Euclidean n-space $\mathbf{R}^n$. We write the equation of C as $x = x(s)$, $0 \leq s \leq L(C)$, where s denotes Euclidean arc length and $L(C)$ is the Euclidean length of C. Denoting differentiation with respect to s by a dot, we define the total curvature of C as

$$K(C) = \int_C |\ddot{x}|\, ds = \int_0^L |\kappa(s)|\, ds, \tag{1}$$

where $\kappa(s)$ denotes the Euclidean curvature parameterized by s. Chakerian [2,3] prove that if C is constrained to lie in a ball of radius R then

$$L(C) \leq RK(C). \tag{2}$$

The curves for which equality holds is (2) are circles of radius R traversed a certain number of times.

In this article we prove the following theorem concerning the p^{th} mean of curvature in Euclidean spaces and consider generalizations of (2) for a closed curve C in a Minkowski plane (Minkowski spaces are simply finite dimensional normed linear spaces).

Theorem 1 *Let C be a close curve of class C^2 in Euclidean n-space R^n. Let $\kappa(s)$ and $K(C)$ denote the curvature and the total curvature of C respectively. Let $L(C)$ denote the length of C. Assume $P \geq 1$ and that C is constrained to lie in a ball of radius R. Then*

$$\left(\int_0^L |\kappa(s)|^P\, ds\right)^{\frac{1}{P}} \geq \frac{L^{\frac{1}{P}}}{R}. \tag{3}$$

curves for which equality holds are circles of radius R traversed a certain number of times.

Preliminary definations and concepts are discussed in section 2. The proof of Theorem 1 and related results are given in section 3.

2. Preliminaries

By a plane convex body we shall mean a compact, convex subset of the Euclidean plane having a non-empty interior. We shall take a "unit circle" E for a Minkowski plane to be a centrally symmetric convex body with its center at the origin in the Euclidean plane. The Minkowski distance defined by E from x to y is given by

$$\|x-y\|_E = \frac{\|x-y\|_e}{r} \tag{4}$$

where $\|x-y\|_e$ is the Euclidean distance from x to y and r is the Euclidean radius of E in the direction of the vector $y-x$. An equivalent definition of Minkowski length is given in [6, p17] by $\|x\|_E = \inf\{\lambda \in \mathbf{R}^+ : x \in \lambda E\}$.

Below we review the definition of the distance function of a convex body K and use the definition to obtain the formula in (4). Let K be a convex body in $\mathbf{R}^n$ containing the origin as an interior point. The distance function of K, $F(K,x)$, is defined by

$$F(K,x) = \inf\{\lambda > 0 : x \in \lambda K\}, x \in \mathbf{R}^n. \tag{5}$$

For a convex body K, the radial function of K in the direction of a unit vector u is defined by

$$r(K,u) = \frac{1}{F(K,u)} \tag{6}$$

Equation (6) and the homogeneity of the distance function imply

$$F(K,x) = F(K, \|x\|_e u) = \|x\|_e F(K,u) = \frac{\|x\|_e}{r(K,u)} \tag{7}$$

where $\|x\|_e$is the usual Euclidean norm and $u = \frac{x}{\|x\|_e}$ for $x \neq 0$ and $\|0\| = 0$. Let E be a centrally symmetric convex body centered at the origin. For each $u \in S^{n-1}$, let $r(E,u)$ be the radius of E in the direction u.

Theorem 2 *Let E be a centrally symmetric convex body in $\mathbf{R}^n$. Define the distance function of E as in* **(5)**. *Let $F(x) = F(K,x)$.* Then $\|\cdot\| = F(\cdot) = \frac{\|x\|_e}{r(K,u)}$ *is a norm.*

Proof. *(i)* $F(x) = 0$ if and only if $x = 0$ is clear.

(ii) If $\lambda > 0$ then $F(\lambda x) = \inf\{\mu > 0 : \frac{\lambda x}{\mu} \in E\} = \lambda \inf\{\mu > 0 : \frac{x}{\mu} \in E\} = \lambda F(x) = |\lambda| F(x)$.

If $\lambda < 0$ then $F(\lambda x) = F((-\lambda)(-x)) = (-\lambda) F(-x) = |\lambda| F(x)$ where $F(-x) = F(x)$ follows from the fact that E is symmetric.

(iii) Since $\frac{x}{F(x)} \in E$ and $\frac{y}{F(y)} \in E$, by the convexity of E we have $\frac{F(x)}{F(x)+F(y)} \cdot \frac{x}{F(x)} + \frac{F(y)}{F(x)+F(y)} \cdot \frac{y}{F(y)} \in E$.Hence $\frac{x+y}{F(x)+F(y)} \in E$.

$$F(x+y) = \inf\{\lambda > 0 : \frac{x+y}{\lambda} \in E\} \leq F(x) + F(y) \tag{8}$$

Assuming that the boundary of the unit circle E has nowhere zero Euclidean curvature, we define the Minkowskian curvature of a curve C at a point P by

$$\kappa_m(C,P) = \frac{\kappa_e(C,P)}{\kappa_e(E,\bar{P})} \tag{9}$$

where $\kappa_e(C,P)$ and $\kappa_e(E,\bar{P})$ denote the Euclidean curvature of C and E at points P and $\bar{P}$ respectively, such that the unit tangent to C at P is parallel to the unit tangent to E at $\bar{P}$. If there are two different points $\bar{P}$ and $\hat{P}$ of E with parallel tangents to the tangent to C at P, then using the fact that E is centrally symmetric, it follows that $\kappa_e(E,\bar{P}) = \kappa_e(E,\hat{P})$. Note that if we use the parametrization Θ for E where Θ is the angle that the tangent to E makes with the horizontal, then $\kappa_e(E,\Theta) = \kappa_e(E,\Theta+\pi)$ because of the fact that E is centrally symmetric.

Other works define Minkowskian curvature differently. See for example Busemann [1]. Our definition permits us to give a generalization of Theorem 1. Note that in the Euclidean case $\kappa_e(E,) = 1$ and the Minkowskian curvature is the same as the usual Euclidean curvature. In order to define the Minkowskian length of a curve, we need the following concepts, which can be found in Conway [4, p.58].

A function $\gamma : [a,b] \to \mathbf{R}^n$ is of bounded variation on $[a,b]$ with respect to the norm $\|\cdot\|$ if and only if there is a constant $M > 0$ such that for any partition

$P = \{a = t_0 < t_1 < t_2 < ... < t_n = b\}$ of $[a,b]$, the total variation of γ over $[a,b]$ satisfies

$$V(\gamma,P) = \sum_1^n \|\gamma(t_k) - \gamma(t_{k-1})\| \leq M. \tag{10}$$

Theorem 3 *A function $\gamma : [a,b] \to \mathbf{R}^n$ is of bounded variation with respect to the Euclidean norm $\|\cdot\|_e$ if and only if it is of bounded variation with respect to the Minkowskian norm $\|\cdot\|$.*

Proof. Let R_1 and R_2 be the minimum and maximum radii of the unit ball E measured in the Euclidean norm. Let $P = \{a = t_0 < t_1 < t_2 < ... < t_n = b\}$ be a partition of $[a,b]$. Let $V_e(\gamma,P)$ and $V_m(\gamma,P)$ denote the total variations with respect to the Euclidean and Minkowskian norms, respectively, and let u_k be a unit vector in the direction of $\gamma(t_k) - \gamma(t_{k-1})$.

Suppose γ is of bounded variation with respect to the Euclidean norm. We can find a bound for $V_m(\gamma,P)$ as follows:

$$V_m(\gamma,P) = \sum_1^n \|\gamma(t_k) - \gamma(t_{k-1})\| = \sum_1^n \frac{\|\gamma(t_k) - \gamma(t_{k-1})\|_e}{r(E,u_k)} \leq \frac{V_e(\gamma,P)}{R_1} \tag{11}$$

If γ is of bounded variation with respect to the Minkowskian norm $\|\cdot\|$, then

$$\begin{aligned} V_e(\gamma,P) &= \sum_1^n \|\gamma(t_k) - \gamma(t_{k-1})\|_e \\ &= \sum_1^n r(E,u_k)\, \|\gamma(t_k) - \gamma(t_{k-1})\| \leq R_2 V_m(\gamma,P). \end{aligned} \tag{12}$$

A path is a continuous function $\gamma : [a,b] \to \mathbf{R}^n$. γ is a rectifiable path if and only if γ is a function of bounded variation. Theorem 3 shows that a path is rectifiable with respect to the Euclidean norm if and only if it is rectifiable with respect to the Minkowskian norm.

Let P be a polygonal path with vertices $x_0, x_1, x_2, \ldots, x_n$. The Minkowskian length of P with respect to the unit ball E is defined by

$$\mu_E(P) = \sum_1^n \|x_i - x_{i-1}\|, \tag{13}$$

where $\|\cdot\|$ is the Minkowskian norm.

Let C be a rectifiable path. The Minkowskian length of C with respect to the unit ball E is defined by

$$\mu_E(C) = \sup_{P \in \prod} \mu_E(P) \tag{14}$$

where $\prod$ is the set of all polygonal paths inscribed in C.

Assume now that C is continuously differentiable. If we let $ds(C,u)$, $u \in S^{n-1}$, denote the Euclidean element of arc length at a point where the tangent vector to the curve C has direction u, then the Minkowskian element of arc length, denoted by $ds_m(C,u)$, is defined by

$$ds_m = \frac{ds_e(C,u)}{r(E,u)}. \tag{15}$$

We can use (13), (14) and (15) to find the following expression for the Minkowskian length of C, denoted $l(C)$.

$$l(C) = \mu_E(C) = \int_C \frac{ds_e(C,u)}{r(E,u)} = \int_C ds_m(C,u). \tag{16}$$

Turning to the case of curves in $\mathbf{R}^2$, we let $ds(C,u) = ds(C,\theta)$, where $u = (\cos\theta, \sin\theta)$. The *self-circumference* of the unit circle E is the Minkowskian length of E measured with respect to E and is given by

$$\mu_E(\partial E) = \int_0^{2\pi} \frac{ds_e(E,\theta)}{r(E,\theta)}. \tag{17}$$

Results In this section, we prove Theorem 1 concerning the p^{th} mean of curvature in Euclidean spaces and consider generalization of (1) for a closed curve C in a Minkowski plane.

Proof of Theorem 1 We can use Hölder's inequality to write

$$K(C) = \int_0^L 1\,|\kappa(s)|\,ds \leq \left(\int_0^L 1 ds\right)^{\frac{1}{q}} \left(\int_0^L |\kappa(s)|^p\,ds\right)^{\frac{1}{p}} \tag{18}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$. Multiplying both sides of (18) by R and using (2) we obtain

$$L \leq RL^{p-1\,/\,p} \left(\int_0^L |\kappa(s)|^p\,ds\right)^{1/p}, \tag{19}$$

giving the desired inequality (3). Equality holds if and only if equality holds in (18) and (2). For equality to hold in (18), $|\kappa(s)|^p$ has to be constant. The case of equality in (2) implies that equality holds in (3) if and only if C is a circle of radius R transversed a certain number of times.

The following theorem is a generalization of (2) in a Minkowski plane. We emphasize that $K(C)$ is the total Euclidean curvature of a closed curve C. The proof given here is similar to Chakerian [3] for the inequality (2).

Theorem 4 *Consider a C^2 closed curve C in a Minkowski space with the unit ball E. Let s and s_m denote Euclidean and Minkowskian arc lengths respectively. Let $L(C)$ and $l(C)$ denote Euclidean and Minkowskian lengths of C. Let $x = x(s)$, $0 \le s \le L(C)$ be a parametric representation of C. Let "dot" denote differentiation with respect to s. Then*

$$l(C) \le RK(C) + \int_C \left|\frac{\dot{r}}{r}\right| ds, \tag{20}$$

where $K(C)$ denotes the total Euclidean curvature of C, r is the radius of E in the direction of arc length ds, and C is constrained to lie in a copy of E with a dialation factor of R.

Proof. The Minkowskian arc length is given by $ds_m = \dfrac{ds}{r}$. Since s is Euclidean arc length, $\dot{x}(s) \cdot \dot{x}(s) = 1$. Hence we can write $l(C)$ as follows

$$l(C) = \int_C ds_m = \int_C \frac{ds}{r} = \int_C \frac{1}{r}\dot{x} \cdot \dot{x} ds. \tag{21}$$

Using integration by parts and the triangle inequality, we obtain

$$\begin{aligned} l(C) &= x \cdot \frac{\dot{x}}{r}\Big|_0^{L(C)} - \int_c x \cdot \left(\frac{\dot{x}}{r}\right) ds \\ &= -\int_C x \cdot \left(\frac{\dot{x}}{r} ds\right) \le \int \|x\|_e \left\|\frac{\ddot{x}r - \dot{r}\dot{x}}{r^2}\right\|_e ds \end{aligned} \tag{22}$$

The fact that C lies in a copy of E magnified by dialation factor of R implies

$$\|x\|_e \le Rr. \tag{23}$$

Using (22) and (23) we obtain

$$\begin{aligned} l(C) &\le \int_C \|x\|_e \left\|\frac{\ddot{x}r - \dot{r}\dot{x}}{r^2}\right\|_e ds \\ &\le \int_C Rr \left\|\frac{\ddot{x}r - \dot{r}\dot{x}}{r^2}\right\| ds \\ &\le \int_C |\ddot{x}| ds + \int_C \left\|\frac{\dot{r}\dot{x}}{r}\right\|_e ds = RK(C) + \int_C \left|\frac{\dot{r}}{r}\right| ds. \end{aligned}$$

In Euclidean spaces the radius of the unit ball in each direction is constant and $\dot{r} = 0$. Hence (20) implies (2).

Fenchel's theorem states that the total curvature of a closed space curve C is greater than or equal to 2π. It is equal to 2π if and only if C is a plane convex curve. See Chern [5] for an elementary discussion of Fenchel's theorem. The following theorem shows that the total Minkowskian curvature of a closed convex curve is equal to the self-circumference of the unit circle E.

Theorem 5 *Let C be a closed convex curve in a Minkowski plane with the unit circle E. Then the total Minkowskian curvature of C is equal to the self-circumference of the unit ball E.*

Proof. Let θ be the angle between the tangent to C and the horizontal. Then,

$$\int_C \frac{\kappa(C,\theta)}{\kappa(E,\theta)} ds_m(C,\theta) = \int_C \frac{\kappa(C,\theta)}{\kappa(E,\theta)} \cdot \frac{1}{r(E,\theta)} ds(C,\theta)$$

$$= \int_C \frac{d\theta}{\kappa(E,\theta)r(E,\theta)} = \int_C \frac{R(E,\theta)d\theta}{r(E,\theta)}$$

$$= \int \frac{ds(E,\theta)}{r(E,\theta)} = L(E),$$

where we have used the fact that $ds = \kappa(C,\theta)ds(C,\theta)$ and that $R(E,\theta)d\theta = ds(E,\theta)$.

References

[1] **H. Busemann**, *The foundations of Minkowskian geometry,* Comm. Math. Helv., 24, 1950, pp.156-187.

[2] **G. D. Chakerian**, *An inequality for closed space curves*, Pacific J. Math. 12, (1962), pp. 53-57.

[3] **G. D. Chakerian**, *On some geometric inequalities*, Proc. Amer. Math. Soc., 15, (1964), pp. 886-888.

[4] **J. B. Conway**, *Functions of one complex variable*, Graduate Texts in Mathematics 11, Springer, 1973.

[5] **S. S. Chern**, *Curves and surfaces in Euclidean space*, Studies in Global Geometry and Analysis, Studies in Math., vol. 4, MAA, 1967.

[6] **A. C. Thompson**, *Minkowski Geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, 1996.