

## RADIAL DISTRIBUTION OF TEMPERATURE IN SPHERICAL BODIES WITH MEMORY

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### Abstract

The aim of this note is to present new solutions in the linearized theory of heat conduction with memory. To this end, by means of the integral transforms, the axis-symmetrical distribution of the temperature in two spherical bodies is established.

### 1. Introduction

The classical linear theory of heat conduction has two principal draw backs. First, it is unable to take into account the memory effects which may be prevalent in some materials and secondly, it leads to an unrealistic result: that a thermal disturbance has an infinite speed of propagation. In order to remove these inconveniences, different theories have been proposed and studied [2,7-9,11]. The most general of them seems to be that of Nunziato [11] in which the full history of the temperature  $\theta(x, t)$  and of its spatial gradient influences the material response at time  $t$ .

In the linear case, this theory, for isotropic media, is based on the linearized heat equation [11]

$$\alpha(0)\dot{\theta}(\mathbf{x}, t) + \int_0^\infty \alpha'(s)\dot{\theta}(\mathbf{x}, t-s)ds = k(0)\nabla^2\theta(\mathbf{x}, t) + \int_0^\infty \nabla^2\theta(\mathbf{x}, t-s)ds + r(\mathbf{x}, t) \quad (1.1)$$

where  $\alpha(\cdot)$  is the energy-temperature relaxation function,  $\alpha(0)(> 0)$  is the instantaneous heat capacity,  $k(\cdot)$  is the heat conduction relaxation function,  $k(0) (\geq 0)$  is the instantaneous conductivity,  $r(\mathbf{x}, \cdot)$  is the heat supply and the dots above letters and primes are used to indicate the partial derivatives with respect to  $t$  and  $s$ , respectively.

### 2. Distribution of temperature in spherical bodies

The equation (1.1), together with suitable initial and boundary conditions, has been investigated in [1,4,11] where different theorems of existence and uniqueness were proved and in [5,6] where some approximate solutions have been obtained.

Our purpose here is to find its solutions in two specific cases. More precisely, using the finite Hankel transforms, these solutions are given as Fourier-Bessel series, whose coefficients, as functions of  $t$ , satisfy to some linear Volterra integral equations.

The solutions of these integral equations, in suitable conditions, can be estimated as closely as possible by means of the successive approximations [3,10] or represented with the aid of the associated resolvent kernel [3].

### 3. The distribution of the temperature in a hollow sphere

Let  $B$  be a rigid heat conductor with memory bounded by two concentric spheres of radii  $R_1$  and  $R_2 (> R_1)$ . Its temperature, till the moment  $t = 0$ , is assumed to be given by the function  $\theta_0(r, t)$ , where  $r$  is the radial spherical co-ordinate. After this moment, the two surfaces of the body are maintained at the uniform temperatures  $\theta_1(t)$  and  $\theta_2(t)$ , so as to

$$\lim_{r \searrow R_1, t \nearrow 0} \theta_0(r, t) = \lim_{t \searrow 0} \theta_1(t), \quad \lim_{r \nearrow R_2, t \nearrow 0} \theta_0(r, t) = \lim_{t \searrow 0} \theta_2(t)$$

Due to the symmetry, we can assume that the temperature of the body will vary after  $r$  and  $t$  only. In this case, in order to determine the distribution of the temperature in the body, we have to solve the next integro-differential equation (see (1.1))

$$\begin{aligned} \alpha(0)\dot{\theta}(r, t) + \int_0^t \dot{\alpha}(t-s)\theta'(r, s)ds &= k(0) \left( \partial_r^2 + \frac{2}{r}\partial_r \right) \theta(r, t) + \\ + \int_0^t \dot{k}(t-s) \left( \partial_r^2 + \frac{2}{r}\partial_r \right) \theta(r, s)ds + f(r, t); & \quad r \in (R_1, R_2), \quad t > 0 \\ + \int_0^t \dot{k}(t-s) \left( \partial_r^2 + \frac{2}{r}\partial_r \right) \theta(r, s)ds + f(r, t); & \quad r \in (R_1, R_2), \quad t > 0 \end{aligned} \quad (2.1)$$

with initial and boundary conditions

$$\theta(r, 0) = \theta_0(r, 0); \quad r \in [R_1, R_2] \quad (2.2)$$

respectively

$$\theta(R_1, t) = \theta_1(t), \quad \theta(R_2, t) = \theta_2(t); \quad t \geq 0 \quad (2.3)$$

in which

$$f(r, t) = r(r, t) - \int_{-\infty}^0 \dot{\alpha}(t-s)\theta'_0(r, s)ds + \int_{-\infty}^0 \dot{k}(t-s) \left( \partial_r^2 + \frac{2}{r}\partial_r \right) \theta_0(r, s)ds$$

is a known function.

Making the change of unknown function

$$\theta(r, t) = r^{-1/2}y(r, t) + \frac{(r - R_1)\theta_2(t) - (r - R_2)\theta_1(t)}{R_2 - R_1} \quad (2.4)$$

our problem reduces to

$$\alpha(0)\dot{y}(r, t) + \int_0^t \dot{\alpha}(t-s)y'(r, s)ds = \quad (2.5)$$

$$= k(0)Ly(r, t) + \int_0^t \dot{k}(t-s)Ly(r, s)ds + g(r, t); \quad r \in (R_1, R_2), \quad t > 0$$

$$y(r, 0) = y_0(r); \quad r \in [R_1, R_2] \quad (2.6)$$

and

$$y(R_1, t) = y(R_2, t) = 0; \quad t \geq 0 \quad (2.7)$$

where the operator  $L = \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{4r^2}$ ,

$$g(r, t) = r^{1/2} \left\{ f(r, t) + \alpha(0) \frac{(r - R_2)\dot{\theta}_1(t) - (r - R_1)\dot{\theta}_2(t)}{R_2 - R_1} + \right. \\ \left. + \frac{1}{(R_2 - R_1)} \int_0^t \dot{\alpha}(t - s) [(r - R_2)\theta'_1(s) - (r - R_1)\theta'_2(s)] ds \right\} + \\ + \frac{2}{(R_2 - R_1)r^{1/2}} \left\{ k(0) [\theta_2(t) - \theta_1(t)] + \int_0^t \dot{k}(t - s) [\theta_2(s) - \theta_1(s)] ds \right\}$$

and

$$y_0(r) = \left[ \theta_0(r, 0) + \frac{(r - R_2)\theta_1(0) - (r - R_1)\theta_2(0)}{R_2 - R_1} \right] r^{1/2}$$

In order to find the solution of the problem (2.5)-(2.7) we multiply both sides of the equation (2.5) by  $rB_{1/2}(rr_n)^1$  and integrate over  $r$  between the limits  $R_1$  and  $R_2$ . Taking into account (2.6), (2.7) and the relation (98.14) from [12], we attain to

$$\alpha(0)\dot{y}_n(t) + \int_0^t \dot{\alpha}(t - s)y'_n(s)ds = -r_n^2 k(0)y_n(t) - r_n^2 \int_0^t \dot{k}(t - s)y_n(s)ds + g_n(t); \quad t > 0 \quad (2.8)$$

and

$$y_n(0) = y_{0n} \quad (2.9)$$

where  $y_n(\cdot)$ ,  $g_n(\cdot)$  and  $y_{0n}$  are the finite Hankel transforms of the functions  $y(r, \cdot)$ ,  $g(r, \cdot)$  and  $y_0(r)$ , respectively.

The equation (2.8), after an integration by parts in the integral of its left side, can be written in the equivalent form

$$\dot{y}_n(t) = \alpha_n y_n(t) + \int_0^t \beta_n(t - s)y_n(s)ds + h_n(t) \quad (2.10)$$

where  $\alpha_n = -\frac{\dot{\alpha}(0) + r_n^2 k(0)}{\alpha(0)}$ ,  $\beta_n(\cdot) = -\frac{\dot{\alpha}(\cdot) + r_n^2 \dot{k}(\cdot)}{\alpha(0)}$  and  $h_n(\cdot) = \frac{\dot{\alpha}(\cdot)y_{0n} + g_n(\cdot)}{\alpha(0)}$ .

Finally, integrating (2.10) between the limits 0 and  $t$ , namely

$$y_n(t) = y_{0n} + \alpha_n \int_0^t y_n(s)ds + \int_0^t \int_0^\tau \beta_n(\tau - s)y_n(s)d\tau ds + \int_0^t h_n(s)ds$$

and taking into account the fact that the iterated integral can be considered as a double integral over the domain

$$\{(\tau, s); 0 \leq \tau \leq t, 0 \leq s \leq \tau\} \equiv \{(s, \tau); 0 \leq s \leq t, s \leq \tau \leq t\}$$

we attain to the next linear Volterra integral equation of convolution type

$$y_n(t) = \gamma_n(t) + \int_0^t \delta_n(t - s)y_n(s)ds; \quad y_n(0) = y_{n0} \quad (2.11)$$

in which  $\gamma_n(t) = y_{0n} + \int_0^t h_n(s)ds$  and  $\delta_n(t) = \alpha_n + \int_0^t \beta_n(s)ds$ .

<sup>1</sup> Here  $B_{1/2}(rr_n) = J_{-1/2}(rr_n)J_{1/2}(R_1 r_n) - J_{-1/2}(R_1 r_n)J_{1/2}(rr_n)$  where  $J_{1/2}(\cdot)$  and  $J_{-1/2}(\cdot)$  are Bessel functions of the first kind and  $r_n = n\pi/(R_2 - R_1)$  are roots of the transcendental equation  $B_{1/2}(R_2 r) = 0$ .

For each fixed  $n$ , the equation (2.11) has an unique solution (see for example [3] or [10]) on  $[0, \infty)$  if  $\gamma_n(\cdot)$  is continuous on  $[0, \infty)$  and  $\delta_n(\cdot)$  is measurable on every finite subinterval  $[0, T]$  of  $(0, \infty)$ . This solution can be represented by the formula ([3] § 4.2)

$$y_n(t) = \gamma_n(t) + \int_0^t \bar{\delta}_n(t-s)\gamma_n(s)ds \quad (2.12)$$

where  $\bar{\delta}(t-s)$  is the resolvent kernel associated with  $\delta_n(t-s)$  and it can be estimated as closely as possible by means of the Picard or Charatheodory successive approximations. As regards our solution, having the relation (2.4) and the inversion theorem for finite Hankel transform ([12] Sect. 97) in mind, it has the form

$$\theta(r, t) = \frac{(r - R_1)\theta_2(t) - (r - R_2)\theta_1(t)}{R_2 - R_1} + \frac{\pi^2}{2r^{1/2}} \sum_n \frac{r_n^2 J_{1/2}^2(R_2 r_n)}{J_{1/2}^2(R_1 r_n) - J_{1/2}^2(R_2 r_n)} B_{1/2}(r r_n) y_n(t) \quad (2.13)$$

where the summation extends over all  $n \in \mathbb{N}^*$ .

#### 4. The case of a full sphere

Let us now consider a sphere of radius  $R$  whose temperature, till the moment  $t = 0$ , is given by the same function  $\theta_0(r, t)$ . After this moment, the surface of the sphere  $r = R$  is maintained at the temperature  $\theta_1(t)$ , so as to

$$\lim_{r \nearrow R, t \searrow 0} \theta_0(r, t) = \lim_{t \searrow 0} \theta_1(t)$$

In this case, the conditions (2.2) and (2.3) have to be changed by

$$\theta(r, 0) = \theta_0(r, 0); \quad r \in [0, R] \quad (3.1)$$

$$\theta(R, t) = \theta_1(t); \quad t \geq 0 \quad (3.2)$$

and the natural condition of boundedness in  $r = 0$ , i.e.

$$|\theta(0, t)| < \infty; \quad t \geq 0 \quad (3.3)$$

Making again a change of an unknown function

$$\theta(r, t) = r^{-1/2} y(r, t) + \frac{r}{R} \theta_1(t) \quad (3.4)$$

our problem reduces to

$$\alpha(0)y'(r, t) + \int_0^t \dot{\alpha}(t-s)y'(r, s)ds = k(0)Ly(r, t) + \quad (3.5)$$

$$+ \int_0^t \dot{k}(t-s)Ly(r, s)ds + g(r, t); \quad r \in (0, R), \quad t > 0 \quad (3.6)$$

and

$$y(R, t) = 0; \quad t \geq 0 \quad (3.7)$$

where  $L$  is the same operator as before,  $y_0(r) = r^{1/2}[\theta_0(r, 0) - r\theta_1(0)/R]$  and

$$g(r, t) = r^{1/2} \left\{ f(r, t) - \frac{r}{R} \left[ \alpha(0)\dot{\theta}_1(t) + \int_0^t \dot{\alpha}(t-s)\theta_1'(s)ds \right] + \right. \\ \left. + \frac{2}{r^{1/2}R} \left[ k(0)\theta_1(t) + \int_0^t \dot{k}(t-s)\theta_1(s)ds \right] \right\}.$$

In order to find the solution of this new problem we can follow the same way as before. More exactly, we multiply (3.5) by  $rJ_{1/2}(rr_n)^2$ , integrate the result between the limits 0 and  $R$  and so on.

Finally, using again the inversion theorem for the corresponding finite Hankel transform, we get  $\theta(r, t)$  under the form

$$\theta(r, t) = \frac{r}{R}\theta_1(t) + \frac{2}{R^2r^{1/2}} \sum_n \frac{J_{1/2}(rr_n)}{[J'_{1/2}(Rr_n)]^2} y_n(t) \quad (3.8)$$

where  $y_n(\cdot)$  are solutions of some linear Volterra integral equations of convolution type of the form (2.11).

## 5. Conclusions

The problem of finding the distribution of the temperature in different bodies with memory seems to be an open one. In the present work the distribution of the temperature in the two spherical bodies with memory is given by the relations (2.13) and (3.8) where the functions  $y_n(\cdot)$ , solutions of the integral equations (2.11), can be estimated as closely as possible by means of the Picard or Charatheodory successive approximations. They can be also represented by the integral formula (2.12) in terms of the associate resolvent kernels.

The extension of these results to more general cases can be easily realised. So, in order to determine the distribution of the temperature in a conical wedge with memory, bounded by the surfaces  $r = R$  and  $\vartheta = k$  (a constant angle), we can use both the finite Hankel transform and a well known developing theorem of Steklov [13]. In view of this theorem the solution of the problem  $\theta(r, \vartheta, t)$ , whose existence it is assumed, can be written for each  $t > 0$  and  $r \in (0, R)$  as a Fourier series absolutely and uniformly convergent in terms of the eigenfunctions of an associate boundary problem.

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<sup>2</sup> Now,  $r_n = n\pi/R$  are roots of the transcendental equation  $J_{1/2}(Rr) = 0$

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