

CONTROLLABILITY OF SECOND ORDER SEMILINEAR DELAY INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

K. BALACHANDRAN and S. MARSHAL ANTHONI

Abstract

Sufficient conditions for controllability of semilinear second order delay integrodifferential systems in Banach spaces are established. The results are obtained by using the theory of strongly continuous cosine families and the Schaefer fixed point theorem.

Key words: Controllability, Semilinear integrodifferential systems, Schaefer theorem.

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1. Introduction

Controllability of linear and nonlinear infinite dimensional systems in Banach spaces has been studied by many researchers. Chukwu and Lenhart [5] have studied the controllability of nonlinear systems in abstract spaces. Naito [7,8] has studied the controllability for semilinear systems and nonlinear Volterra Integrodifferential systems. Quinn and Carmichael [12] have shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. Balachandran et al. [1,2] established sufficient conditions for controllability of nonlinear integrodifferential systems in Banach spaces.

In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. A useful tool for the study of abstract second order equations is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory [14,15]. Motivation for second order systems can be found in [3,6,16]. Recently, Park and Han [11] have discussed the controllability of second order nonlinear systems in Banach spaces using the Schauder fixed point theorem. The purpose of this paper is to study the controllability of semilinear second order delay integrodifferential systems in Banach spaces by using the Schaefer theorem.

2. Preliminaries

We consider the semilinear second order delay control system of the form

$$\begin{aligned} x''(t) &= A(t) + f\left(t, x(\sigma_1(t)), \int_0^t h(t,s)g(s, x(\sigma_2(s)), x'(\sigma_3(s)))ds, x'(\sigma_4(t))\right) + Bu(t), \\ x(0) &= x_0, \quad x'(0) = y_0, \quad t \in J = [0, T], \end{aligned} \quad (1)$$

where the state $x(\cdot)$ takes values in the Banach space X , $x_0, y_0 \in X$, A is a linear infinitesimal generator of the strongly continuous cosinefamily $C(t)$, $t \in R$, of bounded linear operators in X , $g : J \times X \times X \rightarrow X$, $h : J \times J \rightarrow R$, $f : J \times X \times X \times X \rightarrow X$ are given functions, B is a bounded linear operator from U to X and the control function $i(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U as a Banach space. Moreover $\sigma_i : I \rightarrow I$, $i = 1, 2, 3, 4$ are continuous functions such that $\sigma_i(t) \leq t$, $i = 1, 2, 3, 4$.

Definition 1. [14] A one parameter family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff

- (i) $C(s+t) + C(s-t) = 2C(s)C$ for all $s, t \in R$;
- (ii) $C(0) = I$;
- (iii) Cx is continuous in t on R for each fixed $x \in X$.

Define the associated sine family $S(t)$, $t \in R$, by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in R.$$

Assume the following conditions on A .

A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from X into itself and the adjoint operator A^* is densely defined i.e. $\overline{D(A^*)} = X^*$ (See [4]).

The infinitesimal generator of a strongly cosine family $C(t)$, $t \in R$, is the operator $A : X \rightarrow X$ defined by

$$Ax = \left. \frac{d^2}{dt^2} C(t)x \right|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in A : C(t)x \text{ is twice continuously differentiable in } t\}$.

Define $E = \{x \in X; C(t)x \text{ is once continuously differentiable in } t\}$.

To establish our main theorem we need the following lemmas.

Lemma 1. [14] Let (H_1) hold. Then

- (i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Ne^{\omega|t|} \text{ and } \|S(t) - S(t^*)\| \leq N \left| \int_t^{t^*} e^{\omega|s|} ds \right| \text{ for } t, t^* \in R;$$

- (ii) $S(t)X \subset E$ and $S(t) \subset D(A)$ for $t \in R$;
- (iii) $\frac{d}{dt} C(t)x = AS(t)x$ for $x \in E$ and $t \in R$;
- (iv) $\frac{d^2}{dt^2} C(t)x = AC(t)x$ for $x \in D(A)$ and $t \in R$.

Lemma 2. [14] Let (H_1) hold, let $v : R \rightarrow X$ such that v is continuous and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then

$$q \text{ is twice continuously differentiable and for } t \in R, q \in D(A),$$

$$q'(t) = \int_0^t C(t-s)v(s)ds, \text{ and } q''(t) = Aq(t) + v(t).$$

Lemma 3. (Schaefer Theorem) [13] Let S be a convex subset of a normal linear space Y and assume $0 \in S$. Let $F : S \rightarrow S$ be a completely continuous operator, and let

$$\zeta(F) = \{x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

We make the following assumptions:

(H₂) $g : J \times X \times X \rightarrow X$ is continuous in t and the function $h : J \times J \rightarrow R$ is measurable.

(H₃) There exists a continuous function $n : J \rightarrow [0, \infty)$ such that

$$\|g(t, x, y)\| \leq n(t)\Omega(\|x\| + \|y\|), \quad t \in J, \quad x, y \in X,$$

where $\Omega : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function.

(H₄) There exists a constant L such that $\|h(t, s)\| \leq L$, for $t, s \in J$.

(H₅) $f(t, \cdot, \cdot, \cdot) : X \times X \times X \rightarrow X$ is continuous for each $t \in J$, and the function $f(\cdot, x, y, z) : J \rightarrow X$ is strongly measurable for each $x, y, z \in X$.

(H₆) For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|, \|y\|, \|z\| \leq k} \|f(t, x, y, z)\| \leq \alpha_k(t) \text{ for a.a. } t \in J.$$

(H₇) There exists a continuous function $m : J \rightarrow [0, \infty)$ such that

$$\|f(t, x, y, z)\| \leq m(t)\Omega_0(\|x\| + \|y\| + \|z\|), \quad t \in J, \quad x, y, z \in X,$$

where $\Omega_0 : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function and

$$\int_0^T \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}$$

where $\hat{m}(t) = \max\{M(T+1)m(t), Ln(t)\}$,

$$M = \sup\{\|C(t)\| : t \in J\}, \text{ and } M^* = \sup\{\|AS(t)\| : t \in J\},$$

and

$$c = (M + M^*)\|x_0\| + (1+T)M\|y_0\| + (1+T)MT\|B\|\|W^{-1}\|$$

$$\left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T m(s)\Omega_0 \left(\|x(s)\| + \|x'(s)\| + L \int_0^s n(\tau)\Omega(\|x(\tau)\| + \|x'(\tau)\|)d\tau \right) ds \right].$$

(H₈) $Bu(t)$ is continuous.

(H₉) The linear operator $W : L^2(J, U) \rightarrow X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has a bounded invertible operator $W^{-1} : X \rightarrow L^2(J, U) / \ker W$.

(H₁₀) $C(t)$, $t > 0$ is compact.

Then the system (1) has a mild solution of the form (see [9])

$$\begin{aligned} x(t) &= C(t)x_0 + S(t)y_0 \\ &+ \int_0^t S(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \\ &+ \int_0^t S(t-s)Bu(s)ds. \end{aligned} \quad (2)$$

Definition 2. The system (1) is said to be controllable on J if for every $x_0 \in D(A)$, $y_0 \in E$ and $x_1 \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

3. Main result

Theorem. Suppose (H₁)–(H₁₀) hold. Then the system (1) is controllable on J .

Proof. Using the assumption (H₉), for an arbitrary function $x(\cdot)$ we define the control

$$u(t) = W^{-1} \left[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-s) f\left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \right] (t).$$

Using this control we will show that the operator defined by

$$\begin{aligned} (Fx)(t) &= \\ &C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \\ &+ \int_0^t S(t-s)BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \\ &- \int_0^T S(T-\theta)f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta] (s)ds, \quad t \in J, \end{aligned}$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(T) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time T , provided we obtain a fixed point of the nonlinear operator F .

Consider the space $Z = C^1(J, X)$ with norm

$$\|x\|^* = \max\{\|x\|, \|x'\|\}.$$

In order to study the controllability problem for the system (1), we apply Lemma 3 to the following system as in [9,10]

$$\begin{aligned} x''(t) &= \lambda Ax(t) + \lambda f\left(t, x(\sigma_1(t)), \int_0^t h(t, s)g(s, x(\sigma_2(s)), x'(\sigma_3(s)))ds, x'(\sigma_4(t))\right) \\ &+ \lambda Bu(t), \quad t \in J, \quad \lambda \in (0, 1) \end{aligned} \quad (3)$$

Let x be a mild solution of the system (3). Then from

$$\begin{aligned} x(t) = & \lambda(C(t)x_0 + S(t)y_0) + \lambda \int_0^t S(t-s) \\ & f\left(t, x(\sigma_1(t)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \\ & + \lambda \int_0^t S(t-s)BW^{-1} \left[x_1 - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\theta) \right. \\ & \left. f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta \right] (s)ds \end{aligned}$$

we have

$$\begin{aligned} \|x(t)\| \leq & M\|x_0\| + MT\|y_0\| + MT \int_0^t m(s)\Omega_0 (\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \\ & + L \int_0^s n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau) ds + MT^2\|B\|\|W^{-1}\| [\|x_1\| + M\|x_0\| + MT\|y_0\| \\ & + MT \int_0^T m(s)\Omega_0 (\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| + L \int_0^s n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau) ds] \end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$\|x(t)\| \leq v(t)$$

and

$$v'(t) = MTm(t)\Omega_0 (\|x(\sigma_1(t))\| + \|x'(\sigma_4(t))\| + L \int_0^t n(s)\Omega(\|x(\sigma_2(s))\| + \|x'(\sigma_3(s))\|)ds).$$

But

$$\begin{aligned} x'(t) = & \lambda[AS(t)x_0 + C(t)y_0] \\ & + \lambda \int_0^t C(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \\ & + \lambda \int_0^t C(t-s)BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \\ & - \int_0^T S(T-\theta)f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) d\theta \end{aligned}$$

Thus we have

$$\begin{aligned} \|x'(t)\| \leq & M^*\|x_0\| + M\|y_0\| + M \int_0^t m(s)\Omega_0 (\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| \\ & + L \int_0^s n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau) ds + MT\|B\|\|W^{-1}\| [\|x_1\| + M\|x_0\| + MT\|y_0\| \\ & + MT \int_0^T m(s)\Omega_0 (\|x(\sigma_1(s))\| + \|x'(\sigma_4(s))\| + L \int_0^s n(\tau)\Omega(\|x(\sigma_2(\tau))\| + \|x'(\sigma_3(\tau))\|)d\tau) ds] \end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality, we have

$$\|x'(t)\| \leq r(t)$$

and $r'(t) = Mm\Omega_0 \left(\|x(t)\| + \|x'(t)\| + L \int_0^t n(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \right)$, $t \in J$.

Let

$$w(t) = v(t) + r(t) + L \int_0^t n(s)\Omega(v(s) + r(s))ds, \quad t \in J.$$

Then

$$w(0) = v(0) + r(0) = c, \quad v(t) + r(t) \leq w(t), \quad t \in J,$$

and

$$\begin{aligned} w'(t) &= v'(t) + r'(t) + Ln(t)\Omega(v(t) + r(t)) \\ &\leq MTm(t)\Omega_0(w(t)) + Mm(t)\Omega_0(w(t)) + Ln(t)\Omega(w(t)) \\ &= \hat{m}(t)(\Omega_0(w(t)) + \Omega(w(t))), \quad t \in J. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^T \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J.$$

This inequality implies that there is a constant K such that

$$v(t) + r(t) \leq w(t) \leq K, \quad t \in J.$$

Then

$$\|x\|^* = \max\{\|x\|, \|x'\|\} \leq K,$$

where K depends only on T and on the functions m, n, Ω_0 and Ω .

We shall now prove that the operator $F : X \rightarrow Z$ defined by

$$\begin{aligned} (Fx)(t) &= C(t)x_0 + X(t)y_0 \\ &+ \int_0^t S(t-s)f \left(s, x(\sigma(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \\ &+ \int_0^t S(t-s)BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \\ &- \int_0^T S(T-\theta)f \left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta] (s)ds, \end{aligned}$$

is a completely continuous operator.

Let $B_k = \{x \in Z, \|x\|^* \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned} &\| (Fx)(t_1) - (Fx)(t_2) \| \\ &\leq \| C(t_1) - C(t_2) \| \|x_0\| + \| S(t_1) + S(t_2) \| \|y_0\| \\ &+ \left\| \int_0^{t_1} [S(t_1-s) - S(t_2-s)]f \left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \right\| \\ &+ \left\| \int_{t_1}^{t_2} S(t_2-s)f \left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \right\| \\ &+ \left\| \int_0^{t_1} [S(t_1-s) - S(t_2-s)]BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\ &\left. - \int_0^T S(T-\theta)f \left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta] (s)ds \right\| \end{aligned}$$

$$\begin{aligned}
& + \left\| \int_{t_1}^{t_2} S(t_2 - s)BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\
& - \left. \int_0^T S(T - \theta)f \left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s)ds \Big\| \\
& \leq \|C(t_1) - C(t_2)\| \|x_0\| + \|S(t_1) - S(t_2)\| \|y_0\| \\
& + \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \alpha_k(s) ds + \int_{t_1}^{t_2} \|S(t_2 - s)\| \alpha_k(s) ds \\
& + \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \|B\| \|W^{-1}\| \left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds \\
& + \int_{t_1}^{t_2} \|S(t_2 - s)\| \|B\| \|W^{-1}\| \left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \|(Fx)'(t_1) - (Fx)'(t_2)\| \\
& \leq \|C'(t_1) - C'(t_2)\| \|x_0\| + \|S'(t_1) - S'(t_2)\| \|y_0\| \\
& + \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)] f \left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \right\| \\
& + \left\| \int_0^{t_2} C(t_2 - s) f \left(s, x(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s)) \right) ds \right\| \\
& + \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)]BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\
& - \left. \int_0^T S(T - \theta)f \left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s)ds \Big\| \\
& + \left\| \int_{t_1}^{t_2} C(t_2 - s)BW^{-1} [x_1 - C(T)x_0 - S(T)y_0 \right. \\
& - \left. \int_0^T S(T - \theta)f \left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta)) \right) d\theta \right] (s)ds \Big\| \\
& \leq \|AS(t_1) - S(t_2)\| \|x_0\| + \|C(t_1) - C(t_2)\| \|y_0\| \\
& + \int_0^{t_1} \|C(t_1 - s) - C(t_2 - s)\| \alpha_k(s) ds + \int_{t_1}^{t_2} \|C(t_2 - s)\| \alpha_k(s) ds \\
& + \int_0^{t_1} \|C(t_1 - s) - C(t_2 - s)\| \|B\| \|W^{-1}\| \left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds \\
& + \int_{t_1}^{t_2} \|C(t_2 - s)\| \|B\| \|W^{-1}\| \left[\|x_1\| + M\|x_0\| + MT\|y_0\| + MT \int_0^T \alpha_k(\theta) d\theta \right] ds.
\end{aligned}$$

The right-hand sides are independent of $y \in B_k$ and tends to zero as $t_1 \rightarrow t_2$, since $C(t), S(t)$ are uniformly continuous for $t \in J$ and the compactness of $C(t), S(t)$ for $t > 0$ imply the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$.

Thus F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X .

Let $0 < t \leq T$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $x \in B_k$ we define

$$\begin{aligned} (F_\varepsilon x)(t) &= C(t)x_0 + S(t)y_0 \\ &\quad + \int_0^{t-\varepsilon} S(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) ds \\ &\quad + \int_0^{t-\varepsilon} S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta)f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta \end{aligned}$$

Since $C(t), S(t)$ are compact operators, the set $Y_\varepsilon(t) = \{(F_\varepsilon x)(t) : x \in B_k\}$ is precompact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover for every $x \in B_k$ we have

$$\begin{aligned} &\|(Fx)(t) - (F_\varepsilon x)(t)\| \\ &\leq \int_{t-\varepsilon}^t \left\| S(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) \right\| ds \\ &\quad + \int_{t-\varepsilon}^t \|S(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta)f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta \end{aligned}$$

and

$$\begin{aligned} &\|(Fx)'(t) - (F_\varepsilon x)'(t)\| \\ &\leq \int_{t-\varepsilon}^t \left\| C(t-s)f\left(s, x(\sigma_1(s)), \int_0^s h(s,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(s))\right) \right\| ds \\ &\quad + \int_{t-\varepsilon}^t \|C(t-s)BW^{-1}[x_1 - C(T)x_0 - S(T)y_0 \\ &\quad - \int_0^T S(T-\theta)f\left(\theta, x(\sigma_1(\theta)), \int_0^\theta h(\theta,\tau)g(\tau, x(\sigma_2(\tau)), x'(\sigma_3(\tau)))d\tau, x'(\sigma_4(\theta))\right) d\theta \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in X .

It remains to show that $F : X \rightarrow Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \rightarrow x$ in Z . Then there is an integer q such that $\|x_n(t)\| \leq q, \|x'_n(t)\| \leq q$ for all n and $t \in J$, so $\|x(t)\| \leq q, \|x'(t)\| \leq q$ and $x, x' \in Z$. By (H₂)

$$\begin{aligned} &f\left(t, x_n(\sigma_1(t)), \int_0^t h(t,s)g(s, x_n(\sigma_2(s)), x'_n(\sigma_3(s)))ds, x'_n(\sigma_4(t))\right) \\ &\longrightarrow f\left(t, x(\sigma_1(t)), \int_0^t h(t,s)g(s, x(\sigma_2(s)), x'(\sigma_3(s)))ds, x'(\sigma_4(t))\right) \end{aligned}$$

for each $t \in J$ and since

$$\left\| f \left(t, x_n(\sigma_1(t)), \int_0^t h(t, s)g(s, x_n(\sigma_2(s)), x'_n(\sigma_3(s)))ds, x'_n(\sigma_4(t)) \right) - f \left(t, x(\sigma(t)), \int_0^t h(t, s)g(s, x(\sigma_2(s)), x'(\sigma_3(s)))ds, x'(\sigma_4(t)) \right) \right\| \leq 2\alpha_q(t),$$

we have by dominated convergence theorem

$$\begin{aligned} & \|Fx_n - Fx\| = \\ &= \sup_{t \in J} \left\| \int_0^t S(t-s) \left[f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) - f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right] ds \right. \\ & - \int_0^t S(t-s)BW^{-1} \int_0^T \left[f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) - f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \right] d\theta ds \left. \right\| \\ & \leq \int_0^T \left\| S(t-s) \left[f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) - f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right] \right\| ds \\ & + \int_0^T \left\| S(t-s)BW^{-1} \int_0^T \left[f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) - f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \right] d\theta ds \right\| \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \|(Fx_n)' - (Fx)'\| = \\ &= \sup_{t \in J} \left\| \int_0^t C(t-s) \left[f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) - f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right] ds \right. \\ & - \int_0^t C(t-s)BW^{-1} \int_0^T \left[f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) - f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \right] d\theta ds \left. \right\| \\ & \leq \int_0^T \left\| C(t-s) \left[f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) - f \left(s, x_n(\sigma_1(s)), \int_0^s h(s, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(s)) \right) \right] \right\| ds \\ & + \int_0^T \left\| C(t-s)BW^{-1} \int_0^T \left[f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) - f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \right] d\theta ds \right\| \end{aligned}$$

$$f \left(\theta, x_n(\sigma_1(\theta)), \int_0^\theta h(\theta, \tau)g(\tau, x_n(\sigma_2(\tau)), x'_n(\sigma_3(\tau)))d\tau, x'_n(\sigma_4(\theta)) \right) \Big] d\theta \Big\| ds \rightarrow 0$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{x \in Z : x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently by Schaefer's theorem the operator F has a fixed point in Z . This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus the system (1) is controllable on J .

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