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UNIVALENCE CRITERIA FOR HOLOMORPHIC MAPPINGS IN $\mathbf{C}^n$

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Abstract

In this note we give an univalence criterion which contain as particular cases some univalence criteria for holomorphic mappings in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle \cdot, \cdot \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the norm $\|z\| = \langle z, z \rangle$. Let B^n be the open unit ball in $\mathbb{C}^n$, $B^n = \{z \in \mathbb{C} : \|z\| < 1\}$. We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of linear and continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, with the standard operator norm:

$$\|A\| = \sup\{\|Az\| : \|z\| \leq 1\}, \quad a \in \mathcal{L}(\mathbb{C}^n).$$

Let I denotes the identity in $\mathcal{L}(\mathbb{C}^n)$.

Let $H(B^n)$ be the class of holomorphic mappings from B^n into $\mathbb{C}^n$. We say that $f \in H(B^n)$ is locally bihomorphic in B^n if its Fréchet derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j} \right)_{1 \leq j,k \leq n}$$

is nonsingular at each point $z \in B^n$.

The second Fréchet derivative of a mapping $f \in H(B^n)$, denoted by $D^2 f(z)$, $z \in B^n$, is a symmetric bilinear operator from $\mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$. $D^2 f(z)(z, \cdot)$ is a linear operator obtained by restricting $D^2 f(z)$ to $\{z\} \times \mathbb{C}^n$. This linear operator has the matrix reprezentation

$$D^2 f(z)(z, \cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \leq j,k \leq n}.$$

A mapping $v \in H(B^n)$ is called a Schwarz function if $\|v(z)\| \leq \|z\|$, $z \in B^n$. If $f, g \in H(B^n)$ we say that f is subordinate to g in B^n ($f \prec g$) if there exists a Schwarz function $v(z)$ such that $f(z) = g(v(z))$, $z \in B$.

A family of function $L(z,t)$, $t \geq 0$, is called a subordination chain if $L(\cdot,t)$ is holomorphic and univalent in B^n, $L(0,t) = 0$, for each $t \geq 0$ and $L(z,s) \prec L(z,t)$, whenever $0 \leq s < t < \infty$.

We shall need the following theorem to prove our results.

Theorem 1. [1] *Let* $L(z,t) = a_1(t)z + \ldots$, $a_1(t) \neq 0$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbb{C}^n$ *such that:*

(i) For each $t \geq 0$, $L(\cdot,t) \in H(B^n)$.

(ii) $L(z,t)$ *is a locally absolytely continuous function of* t, *locally uniformly with respect to* $z \in B^n$.

(iii) $a_1(t) \in C^1[0,\infty)$ *and* $\lim\limits_{t\to\infty} |a_1(t)| = \infty$.

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbb{C}^n$ *such that*

(iv) For each $t \geq 0$, $h(\cdot,t) \in H(B^n)$, $h(0,t) = 0$ *and* $\operatorname{Re} \langle h(z,t), z\rangle \geq 0$, $z \in B^n$.

(v) For each $z \in B^n$, $h(z,\cdot)$ *is a measurable function on* $[0,\infty)$.

(vi) For each $T > 0$ *and* $r \in (0,1)$ *there is a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$, *where* $\|z\| \leq r$ *and* $t \in [0,T]$.

Suppose $h(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)h(z,t), \quad a.e. \quad t \geq 0, \quad for\ all \quad z \in B^n. \tag{1}$$

Further, suppose there is a sequence $(t_m)_{m\geq 0}$, $t_m > 0$, $\lim\limits_{m\to\infty} t_m = \infty$ *such that:*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a_1(t_m)} = F(z) \tag{2}$$

locally uniformly in B^n.

Then, for each $t \geq 0$, $L(\cdot,t)$ *is univalent on* B^n.

2. Main results

Theorem 2. *Let* $f, g \in H(B^n)$ *such that* $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ *and* g *is locally biholomorphic in* B^n. *Let* $a : [0,\infty) \to \mathbb{C}$ *be a function which satisfies the conditions:*

1) $a \in C^1[0,\infty)$, $a(0) = 1$ *and* $a(t) \neq 0$, *for all* $t \in [0,\infty)$.

2) $\lim\limits_{t\to\infty} |a(t)| = \infty$.

3) $\operatorname{Re} \dfrac{a'(t)}{a(t)} > 0$, *for all* $t \geq 0$.

If

$$\left\| [Dg(z)]^{-1} Df(z) - \frac{1 + a'(0)}{2} I \right\| < \frac{|1 + a'(0)|}{2}, \quad for\ all \quad z \in B^n \tag{3}$$

and

$$\max_{\|z\|=e^{-t}} \Bigg\| \|z\| [Dg(z)]^{-1} Df(z) + (a(t) - \|z\|)[Dg(z)]^{-1} D^2 g(z)(z,\cdot) + \\ + \left[\frac{a(t) - a't(t)}{2} - \|z\| \right] I \Bigg\| < \frac{|a(t) + a'(t)|}{2} \tag{4}$$

for all $t \geq 0$, *then* f *is an univalent function on* B^n.

Proof. We define

$$L(z,t)=f(e^{-t}z)+(a(t)e^{t}-1)e^{-t}Dg(e^{-t}z)(z),\quad t\geq 0,\quad z\in B^n. \tag{5}$$

Since $a_1(t)=a(t)$ it results $a_1(t)\neq 0$, $a_1\in C^1[0,\infty)$ and $\lim\limits_{t\to\infty}|a_1(t)|=\infty$. We have $L(z,t)=a_1(t)z+$(holomorphic term). Thus $\lim\limits_{t\to\infty}\dfrac{L(z,t)}{a(t)}=z$, locally uniform with respect to B^n and (2) holds with $F(z)=z$. Obviously $L(z,t)$ satisfies the absolute continuity requirements of Theorem 1.

Straightforward calculations show that

$$DL(z,t)=e^{-t}Df(e^{-t}z)+(a(t)e^{t}-1)e^{-t}Dg(e^{-t}z)+(a(t)e^{t}-1)D^2g(e^{-t}z)(e^{-t}z,\cdot)=$$

$$=\frac{a(t)+a'(t)}{2}Dg(e^{-t}z)[I-E(z,t)], \tag{6}$$

where, for each fixed $(z,t)\in B^n\times[0,\infty)$, $E(z,t)$ is the linear operator

$$E(z,t)=-\frac{2e^{-t}}{a(t)+a'(t)}[(Dg(e^{-t}z))^{-1}Df(e^{-t}z)-I]-$$

$$-\frac{2(a(t)-e^{-t})}{a(t)+a'(t)}[Dg(e^{-t}z)]^{-1}(e^{-t}z)(e^{-t}z,\cdot)-\frac{a(t)-a'(t)}{a(t)+a'(t)}I.$$

We shall prove that for each $(z,t)\in B^n\times[0,\infty)$, $I-E(z,t)$ is an invertible operator.

For $t=0$,

$$\|E(z,0)\|=\frac{2}{|1+a'(0)|}\left\|[Dg(z)]^{-1}Df(z)-\frac{1+a'(0)}{2}I\right\|<1.$$

For $t>0$, $E(\cdot,t):\overline{B^n}\to\mathcal{L}(\mathbb{C}^n)$ is holomorphic. By using the weak maximum modulus theorem [2], we obtain that $\|E(z,t)\|$ can have no maximum in B^n unless $\|E(z,t)\|$ is of constant value throughout $\overline{B^n}$.

If $z=0$ and $t>0$ we have

$$\|E(0,t)\|=\left|\frac{a'(t)-a(t)}{a'(t)+a'(t)}\right|<1.$$

We also have $\|E(z,t)\|<\max\limits_{\|w\|=1}\|E(w,t)\|$. If we let $u=e^{-t}w$, with $\|w\|=1$, then $\|u\|=e^{-t}$ and form (4) we obtain

$$\max_{\|w\|=1}\|E(w,t)\|=\max_{\|u\|=e^{-t}}\frac{2}{|a(t)+a'(t)|}\Bigg\|\|u\|(Dg(u))^{-1}Df(u)+$$

$$+(a(t)-\|u\|)[Dg(u)]^{-1}D^2g(u)(u,\cdot)+\left(\frac{a(t)-a'(t)}{2}-\|u\|\right)I\Bigg\|<1.$$

Since $\|E(z,t)\|<1$ for all $(z,t)\in B^n\times[0,\infty)$ it results that $I-E(z,t)$ is an invertible operator.

From (5) we obtain

$$\frac{\partial L(z,t)}{\partial t}=\frac{a(t)+a'(t)}{2}Dg(e^{-t}z)[I+E(z,t)](z)=$$

$$=DL(z,t)[I-E(z,t)]^{-1}[I+E(z,t)](z).$$

Hence $L(z,t)$ satisfies the differential equation (1), for all $z\in B^n$ and $t\geq 0$, where

$$h(z,t)=[I-E(z,t)]^{-1}[I+E(z,t)](z).$$

The function $h(z,t)$ satisfies the holomorphy and measurability requirements of Theorem 1 and $h(0,t)=0$. Since

$$\|h(z,t)-z\| = \|E(z,t)(h(z,t)+z)\| \le \|E(z,t)\| \cdot \|h(z,t)+z\| < \|h(z,t)+z\|$$

we have Re $\langle h(z,t), z\rangle \ge 0,\ z \in B^n,\ t \ge 0.$

By using the inequality

$$\|[I-E(z,t)]^{-1}\| \le [1-\|E(z,t)\|]^{-1}$$

we obtain

$$\|h(z,t)\| \le \frac{1+\|E(z,t)\|}{1-\|E(z,t)\|}\|z\|.$$

The conditions of Theorem 1 being satisfied, it follows that the functions $L(z,t),\ t \ge 0$ are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

Remark. Specific choices for $g(z)$ and $a(t)$ give us some known univalence criteria for holomorphic mappings in $\mathbb{C}^n$ [1], [5], [6].

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