LIBERTAS MATHEMATICA, VOL XX (2000)

SOME SIMPLE CRITERIA FOR STARLIKENESS AND CONVEXITY OF ORDER ALPHA

GHEORGHE OROS

Abstract

We find some criteria involving f' and f'' only in determining the starlikeness of order α or convexity of order α of f or of $F = I[f]$, where I is a certain integral operator. For $\alpha = 0$ these criteria were obtained by P.T. Mocanu in [2].

Key words: starlikeness of order α, convex of order α, integral operator
AMS classification: 30C45.

1. Introduction

Let A_n denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + \dots, \quad n \geq 1$$

that are analytic in the unit disc $U = \{z \in \mathbb{C};\ |z| < 1\}$ and let $A = A_1$.

For $\alpha < 1$ let

$$S_n^*(\alpha) = \left\{ f \in A_n :\ Re\frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\}$$

and

$$K_n(\alpha) = \left\{ f \in A_n :\ Re\frac{zf''(z)}{f'(z)} + 1 > \alpha,\ z \in U \right\}.$$

In [2] the following criteria for starlikeness or convexity were obtained.

Theorem A. *Let $c > -1$, let n be a positive integer and let us define*

$$M = M_n(c) = \frac{n + c + 1}{(c+1)\left[\sqrt{(n+c+1)^2 + (c+1)^2} + |c|\right]}.$$

If $f \in A_n$ and

$$|f'(z) - 1| < M$$

then $I_c[f] \in K_n(0)$, where $I_c : A_n \to A_n$ is the integral operator defined by $F = I_c[f]$, with

$$F(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt, \quad z \in U.$$

Theorem B. *If $f \in A_n$ and*

$$|f''(z)| \le \frac{n}{n+1}, \quad \textit{for} \quad z \in U$$

then $f \in K_n(0)$. This result is sharp.

In this paper we extend these results for any $\alpha < 1$.

2. Preliminaries

Let F and G be two analytic functions in U. If G is univalent, then we say that F is subordinate to G, written $F \prec G$, or $F(z) \prec G(z)$, iff $F(0) = G(0)$ and $F(U) \subset G(U)$.

We shall use the following lemma in order to prove our main results.

Lemma. [1] *Let $c > -1$ and let q be convex in U, with $q(0) = 1$ and let $P(z) = 1 + p_n z^n + \dots$ be analytic in U. If*

$$P(z) + \frac{1}{c+1} zP'(z) < q(z)$$

then $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{(c+1)/n}} \int_0^z q(t) t^{\frac{c+1}{n-1}} dt.$$

3. Main results

Theorem 1. *Let $\alpha < 1$, $c > -1$, let n be a positive integer and let us define*

$$M = M(n, c, \alpha) = \frac{(1-\alpha)(n+c+1)}{(c+1)\left[\sqrt{(n+c+1)^2 + (c+1)^2} + |n+\alpha|\right]}. \tag{1}$$

If $f \in A_n$ and

$$|f'(z) - 1| < M, \tag{2}$$

then $I_c[f] \in K_n(\alpha)$, where $I_c : A_n \to A_n$ is the integral operator defined by $F = I_c[f]$, with

$$F(z) = \frac{c+1}{z^c} \int_0^z f(\zeta)\zeta^{c-1} d\zeta = (c+1) \int_0^1 f(tz) t^{c-1} dt, \quad z \in U. \tag{3}$$

Proof. Let$f \in A_n$ satisfy (2), with M given by (1). From (3) we deduce

$$zF' + cF = (c+1)f$$

and

$$zF'' + (c+1)F' = (c+1)f'. \tag{4}$$

If we set $P = F'$, then (2) is equivalent to

$$P(z) + \frac{1}{c+1} zP'(z) = f'(z) \prec 1 + Mz \equiv q(z)$$

and by Lemma we deduce $P \prec Q$, where

$$Q(z) = \frac{c+1}{nz^{\frac{c+1}{n}}} \int_0^z q(\zeta)\zeta^{\frac{c+1}{n}-1} dt = 1 + \frac{(c+1)M}{n+c+1} z.$$

If we let

$$R = \frac{(c+1)M}{n+c+1}, \tag{5}$$

then

$$|P(z) - 1| < R. \tag{6}$$

Since $R < 1$, we deduce $|F'(z) - 1| < 1$, which shows that F is univalent. Therefore, if we put

$$\frac{zF''(z)}{F'(z)} + 1 = (1-\alpha)p(z) + \alpha,$$

then the function $p(z) = 1 + p_n z^n + \ldots$ is analytic in U and from (4) we deduce

$$P[(1-\alpha)p + c + \alpha] = (c+1)f'. \tag{7}$$

Hence (2) becomes

$$|P(z)[(1-\alpha)p(z) + c + \alpha] - (c+1)| < (c+1)M.$$

If $Rep(z) \not> 0$, then there exists $z_0 \in U$ such that $p(z_0) = is$. Therefore, in order to show that (7) implies $Re\, p(z) > 0$, it is sufficient to check the inequality

$$|P(z)[(1-\alpha)p(z) + c + \alpha] - (c+1)| \geq (c+1)M, \tag{8}$$

for all real s and all $z \in U$.

If we let $P = u + iv$, then

$$E = |P[(1-\alpha)is + c + \alpha] - (c+1)|^2 =$$
$$= (1-\alpha)^2(u^2+v^2)s^2 + 2(1-\alpha)(c+1)vs + |(c+\alpha)P - (c+1)|^2.$$

On the other hand, from (6) we deduce

$$|(c+\alpha)P - (c+1)| \geq 1 - \alpha - |c+\alpha|R,$$

hence

$$E \geq (1-\alpha)^2(u^2+v^2)s^2 + 2(1-\alpha)(c+1)vs + [1-\alpha-(c+\alpha)R]^2$$

and the inequality (8) holds if

$$E - (c+1)^2M^2 \geq (1-\alpha)^2(u^2+v^2)s^2 + 2(1-\alpha)(c+1)vs +$$
$$+[1-\alpha-(c+\alpha)R]^2 - (n+c+1)^2R^2 \geq 0$$

and this last inequality holds if

$$\Delta = (1-\alpha)^2(c+1)^2v^2 - (1-\alpha)^2(u^2+v^2)[(1-\alpha-|c+\alpha|^2R)^2 - (n+c+1)^2R^2] \leq 0,$$

i.e. if

$$v^2[(c+1)^2 + (n+c+1)^2R^2 - (1-\alpha-|c+\alpha|R)^2] \leq$$
$$\leq u^2[(1-\alpha-|c+\alpha|R)^2 - (n+c+1)^2R^2]. \tag{9}$$

From (6) we deduce

$$\frac{v^2}{u^2} \leq \frac{R^2}{1-R^2} \tag{10}$$

and a simple calculation, using (1) and (5), yields

$$\frac{R^2}{1-R^2} \leq \frac{(1-\alpha-|c+\alpha|R)^2 - (n+c+1)^2R^2}{(c+1)^2 + (n+c+1)^2R^2 - (1-\alpha-|c+\alpha|R)^2}.$$

Hence, by using (10), the inequality (9) holds and from (8) and (7) we deduce $Re\, p(z) > 0$, which shows that $F \in K_n(\alpha)$. □

Corollary 1. *Let $\alpha < 1$, $c > -1$, and let n be a positive integer.*

If $f \in A_n$ and

$$|f''(z)| < nM$$

where M is defined by (1), then $I_c[f] \in K_n(\alpha)$, where I_c is defined by (3).

If we take $c = 0$ and use the fact that $F \in K_n(\alpha) \Leftrightarrow f \in S_n^*(\alpha)$, then we deduce the following criteria for starlikeness of order α.

Corollary 2. *If $f \in A_n$ and*

$$|f'(z) - 1| < \frac{(1-\alpha)(n+1)}{\sqrt{(n+1)^2+1} + |\alpha|}$$

then $f \in S_n^*(\alpha)$.

Corollary 3. *If $f \in A_n$ and*

$$|f''(z)| \le \frac{(1-\alpha)n(n+1)}{\sqrt{(n+1)^2+1} + |\alpha|}$$

then $f \in S_n^*(\alpha)$.

Corollary 4. *Let $\alpha < 1$, $c > -1$ and let n be a positive integer. If $f \in A_n$ and*

$$|f'(z) - 1| \le \frac{(1-\alpha)(n+1)(n+c+1)}{(c+1)\sqrt{(n+1)^2+1} + |\alpha|} \tag{11}$$

then $I_c[f] \in S_n^*(\alpha)$, where I_c is defined by (3).

Proof. From (4) and (11) we deduce

$$F' + \frac{1}{c+1} zF'' = f' \prec 1 + \frac{(1-\alpha)(n+1)(n+c+1)}{(c+1)\sqrt{(n+1)^2+1} + |\alpha|} z$$

and by Lemma we obtain

$$F'(z) \prec 1 + \frac{(1-\alpha)(n+1)}{\sqrt{(n+1)^2+1} + |\alpha|} z.$$

Hence

$$|F'(z) - 1| < \frac{(1-\alpha)(n+1)}{\sqrt{(n+1)^2+1} + |\alpha|}$$

and from Corollary 2 we deduce $F \in S_n^*(\alpha)$.

Theorem 2. *If $\alpha < 1$ and if $f \in A_n$ satisfies*

$$|f''(z)| \le M = M_n(\alpha) = \frac{n(1-\alpha)}{n+1-\alpha}, \tag{12}$$

then $f \in K_n(\alpha)$. This result is sharp.

Proof. By Schwarz's lemma we have

$$|f''(z)| \le M|z|^{n-1}.$$

Hence

$$\left|\int_0^z f''(\zeta)d\zeta\right| = \left|z\int_0^1 f''(tz)dt\right| \le M\int_0^1 t^{n-1}dt = \frac{M}{n} = \frac{1-\alpha}{n+1-\alpha}.$$

Since

$$f'(z) = 1 + \int_0^z f''(\zeta)d\zeta,$$

we deduce

$$|f'(z)| \ge 1 - \left|\int_0^z f''(\zeta)d\zeta\right| \ge 1 - \frac{1-\alpha}{n+1-\alpha} = \frac{n}{n+1-\alpha}.$$

Therefore

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1-\alpha$$

which implies $f \in K_n(\alpha)$.

The function

$$f(z) = z + \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}$$

shows that this result is sharp. □

References

[1] **D.J. Hallenbeck, S. Rucheweyh**, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195.

[2] **P.T. Mocanu**, *Some simple criteria for starlikeness and convexity*, Libertas Mathematica (Arlington, Texas), vol.13(1993), 27-40.