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ON AN OPERATOR ALTERNATIVE WITHOUT ODDNESS CONDITIONS

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The note deals with the surjectivity of Fredholm alternative type for nonlinear mappings $\lambda A - T$ on a Banach space, with A invertible, T compact and satisfy homogeneity conditions and $\lambda \in \mathbf{R}$. The interest in this matter is motivated by the recent results obtained by Feng-Webb [1] with lack of oddness conditions.

Let X be a real reflexive space with the dual space X^*, whose norm are $\|\cdot\|$ and $\|\cdot\|_*$. By "$\to$" and "$\rightharpoonup$" we denote the strong and weak convergent, respectively, while $\langle \cdot, \cdot \rangle$ means duality on $X^* \times X$. For $A, T : X \mapsto X^*$ and $\lambda \neq 0$, it is clear that $\lambda A - T$ maps $D(A)$ onto X^* if $I - S$ maps X^* onto X^* where $S ::X^* \to X^*$ is defined by $S(y) = TA^{-1}\left(\frac{y}{\lambda}\right)$. This is the reason to use the measure of noncompactness theory in [1].

Given any bounded set $\Omega \subset X$ define the *Kuratowsky measure of noncompactness* $\alpha(\Omega)$ of Ω as the infimum of those $\varepsilon > 0$ such that Ω can be covered of subsets of Ω having diameter less than or equal to ε. For the properties of $\alpha(\Omega)$ see e.g. [3]. For a continuous map $F : D(F) \subset X \mapsto X^*$ we introduce the following extended real numbers

$$\begin{aligned}
\alpha(F) &= \inf\{k \geq 0 | \alpha(F(\Omega)) \leq k\alpha(\Omega) \text{ for every bounded } \Omega \subset D(F)\}, \\
\beta(F) &= \sup\{k \geq 0 | \alpha(F(\Omega)) \geq k\alpha(\Omega) \text{ for every bounded } \Omega \subset D(F)\}, \\
d(F) &= \lim_{\|x\| \to \infty} \inf_{x \in D(F)} \frac{\|F(x)\|_*}{\|x\|}, \ |F| = \lim_{\|x\| \to \infty} \sup_{x \in D(F)} \frac{\|F(x)\|_*}{\|x\|}
\end{aligned}$$

Here $|F|$ is a quasinorm and F is said *quasibounded* if $|F| < \infty$.

On the other hand, a continuous mapping $K : D(K) \subseteq X \mapsto X^*$ is said to be *compact* if $K(M)$ is relatively compact for bounded subset $M \subseteq D(K)$. Note that a map K satisfies $\alpha(K) = 0$ if and only if K is compact.

DEFINITION 1. *A continuous map $F : X \mapsto X^*$ is said to be stably-solvable if the equation*

$$Fu = Ku$$

has a solution $u \in X$ for any compact map $K : X \mapsto X^*$ with the quasinorm $|K| = 0$.

The map F is *regular* if it is stably-solvable and $d(F)$ and $\beta(F)$ are both positive. The *resolvent* of is the set

$$\rho(S) = \{\lambda \in \mathbf{R} | \lambda \mathbf{I} - S \text{ is regular}\}$$

Finally, if F is invertible, $\alpha(F^{-1}) = \dfrac{1}{\beta(F)}$ so regular invertible maps have k-set contractive inverses.

The basic surjectivity result established in [1] without oddness conditions is

THEOREM 2. *Let $A : D(A) \mapsto X^*$ be an one-to-one operator with $A^{-1} : X^* \mapsto D(A)$ continuous. Suppose there are real positive numbers c and a such that*

$$\|Ax\| \geq c\,\|x\|^{a} \qquad \text{for every } x \in D(A).$$

Let $T : X \mapsto X^*$ be compact and a*-assymptoticaly zero,*

$$\lim_{\|x\|\to\infty} \sup_{x \in D(A)} \frac{\|T(x)\|_*}{\|x\|^{a}} = 0.$$

Then $\lambda A - T$ carries $D(A)$ onto Y for every $\lambda \neq 0$.

Some refinements can be obtained under various homogeneity hypotheses.

An operator $F : X \mapsto X^*$ is said to be *k-homogeneous*, for some $k > 0$, if $F(tu) = t^k Fu$ for all $u \in X$, $t > 0$.

Let F_0 be an k-homogeneous operator. F is said to be *k-quasihomogeneous* with respect to F_0 if $t_n \searrow 0$, $u_n \rightharpoonup u_0$, $t^k F\left(\dfrac{u_n}{t_n}\right) \to g \in Y$ together imply that $g = F_0 u_0$. Finally F is said to be *k-strongly quasihomogeneous* with respect to if F_0 if $t_n \searrow 0$, $u_n \rightharpoonup u_0$, imply that $t^k F\left(\dfrac{u_n}{t_n}\right) \to F_0 u_0$.

A generalization of the concept of stably-solvable map is given by

DEFINITION 3. *A continuous operator $F : D(F) \subset X \mapsto X^*$ is said to be k-stable-solvable if the equation*

$$Fu = Ku$$

has a solution $u \in D(F)$ for any compact map $K : X \mapsto X^*$ with

$$|K|_k = \limsup_{\|u\|\to\infty} \frac{\|Ku\|_*}{\|u\|^{k}}.$$

PROPOSITION 4. *Any operator* A *satisfying the hypotheses of Theorem 2 is* a*-stably-solvable.*

The methodology of surjectivity criteria is based the Continuation Principle [1]:

LEMMA 5. *Let $F : D(F) \subset X \mapsto X^*$ be* a*-stable-solvable and $K : [0,1] \times X \mapsto X^*$ be compact such that $K(0,u) = 0$ for all $u \in D(F)$.* Let

$$U = \{x \in D(F) | Fx = K(t,x) \text{ for some } t \in [0,1]\}.$$

Then, if $F(U)$ is bounded, the equation

$$Fu = K(1,u)$$

has a solution.

Let $A_0, T_0 : X \mapsto X^*$ be k-homogeneous operators. A number λ is called an *eigenvalue* for the couple (A_0, T_0) if there is an element $u \neq 0$ such that

$$\lambda A_0 u - T_0 u = 0 \tag{6}$$

We also call $(\lambda, u) \in \mathbf{R} \times X$ an *eigensolution* for the couple (A_0, T_0).

We can prove the following Fredholm alternative:

THEOREM 7. *Let A be as in Theorem 2 with $D(A) = X$ and also* a*-quasimonotone with respect to A_0. Let $T : X \mapsto X^*$ be a compact* a*-strongly-quasimonotone operator with* *respect to T_0. If $\lambda \neq 0$, and for every $t \in (0,1]$, $\frac{\lambda}{t}$ is not an eigenvalue for the couple (A,T) , then $\lambda A - T$ maps X onto X^*.*

Proof. For arbitrary $f \in X^*$, take $K(t,u) = t(Tu + f)$ and let

$$U = \{u \in X | \lambda Au = t(Tu + f), t \in (0,1]\}$$

We show that U is bounded. If not, there exists $\subset U$, $\|u_n\| \to \infty$, such that

$$\lambda Au_n = t_n (Tu_n + f), \qquad t_n \in [0,1],$$

so that

$$\frac{\lambda Au_n}{\|u_n\|^a} = t_n \left(\frac{Tu_n}{\|u_n\|^a} + \frac{f}{\|u_n\|^a} \right) = t_n \frac{1}{\|u_n\|^a} T \left(\frac{\frac{u_n}{\|u_n\|}}{\frac{1}{\|u_n\|}} \right) + t_n \frac{f}{\|u_n\|^a}.$$

Without loss of generality we assume that $\frac{u_n}{\|u_n\|} \rightharpoonup u_o$, $t_n \to t_0 \in [0,1]$. Passing eventually to a subsequence, we have

$$t_n \frac{1}{\|u_n\|^a} T \left(\frac{\frac{u_n}{\|u_n\|}}{\frac{1}{\|u_n\|}} \right) \to t_0 T_0 u_0 \qquad \text{and thus} \qquad \lim_{n \to \infty} \frac{\lambda Au_n}{\|u_n\|^a} = t_0 T_0 u_0.$$

Since A is a-quasihomogeneous with respect to A_0, we obtain

$$\lambda A_0 u_0 = t_0 T_0 u_0$$

As $\frac{\|\lambda Au_n\|_*}{\|u_n\|^a} \geq |\lambda| c > |\lambda| \frac{c}{2}$ for any n large enough, so that $\|t_0 T_0 u_0\| > 0$. Hence $t_0 \neq 0$ and $T_0 u_0 \neq 0$. Therefore $\frac{\lambda}{t_0}$ and $u_0 \neq 0$ is an eigensolution of (A_0, T_0), a contradiction. Thus U is bounded. So by the Continuation Principle, the equation $\lambda Au = Tu + f$ has a solution $u \in X$, that is $\lambda A - T$ is onto. ■

The above Fredholm alternative under the assumption of oddness of maps A , T is given in [2] and [6]. In this case, if the conditions of Theorem 2 are fulfilled then $\lambda A - T$ is weakly coercive and its surjectivity follows by Borsuk's theorem.

We remain interested in the case when A is of type $(S)_+$ and T is a compact operator. The existence of eigensolutions and Fredholm alternative with A , T odd operators were proved in [8] and with a maximal monotone homogeneous operator A in [4]. In view of this, we may use the continuation method, existence and surjectivity related to the extension of topological degree for mappings of type $(S)_+$, (see e.g. [7]). Finally, similar arguments for a more general nonlinear eigenvalue problem were considered in [5].

References

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