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NON-UNIQUELY EXTREMAL QUASICONFORMAL MAPPINGS

Edgar Reich

"Ce-şi face omul cu mâna lui, nu-i face nici duşmanul lui" [1]

1. Introduction.

For a quasiconformal homeomorphism $f(z)$ of $\Delta = \{|z| < 1\}$ onto itself, we denote as usual the complex dilatation by

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z}, \quad \|\mu_f\|_\infty < 1,$$

and the maximal dilatation by

$$K[f] = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

We denote by f^μ the normalized mapping with complex dilatation μ for which $f^\mu(1) = 1$, $f^\mu(i) = i$, $f^\mu(-1) = -1$. The minimum of $K[g]$ among all quasiconformal mapping s g for which $g|_{\partial\Delta} = f|_{\partial\Delta}$ is denoted by $K_0[f]$. We call f *extremal* (for its boundary values) if $K[f] = K_0[f]$. f is *uniquely extremal* if it is extremal and if there are no other extremal mappings for its boundary values; the alternative is that f is *non-uniquely extremal.*

For reference below, let $L^1_a(\Delta)$ be the set of analytic functions belonging to $L^1(\Delta)$. When $\varphi \in L^1_a(\Delta)$, we denote its L^1-norm over Δ by

$$\|\varphi\| = \iint_\Delta |\varphi|\, dx\, dy.$$

Classically, extremality and unique extremality were established by arguments based on the length-area principle, or its elaboration, extremal length. With the help of this "geometric" method, the first example of non-unique extremality was discovered by Strebel ([9], page 316). By now, numerous additional examples are known, for example, the family

[1] ♣I would like to thank Cornel Constantinescu for teaching me this important Romanian proverb.

of examples considered in [7]. Until recently the proof that there was more than one extremal mapping required a very explicit construction of competing extremals. With the appearance of [2],[3], and [8], a new "analytic" characterization of extremality of a quasiconformal mapping became available in terms of a related problem for the class $L^1_a(\Delta)$, but it did not provide a general way of deciding between unique and non-unique extremality. Historically, the next step were fairly broadly applicable *sufficient* conditions of analytic type for unique extremality given by the author in [5] and [6]. Finally, *necessary and sufficient* conditions of analytic type for unique extremality were found by Božin, Lakic, Marković, and Mateljević [1]. If extremality has already been established, then a necessary condition for unique extremality provides a sufficient condition for non-unique extremality, and somehow its proof must involve an existence proof for additional extremals beyond a given one. In order to better understand the proof in [1], our object is to rewrite it in a more constructive manner in the case when the domain of the mapping is Δ. Namely, the basic construction will make use of an auxiliary mapping that is obtained as a limit of n-gon mappings.

In order to focus on essentials, we restrict ourselves to the technically simplest case, that when the given extremal mapping f has $|\mu_f(z)| \equiv k$. This is actually the most interesting common case, as, if e.g., for some $k' < k$, the set $\{z \in \Delta : |\mu_f(z)| \leq k'\}$ contained interior points, the fact that f is not the only extremal mapping is trivial. The relevant theorem of [1], specialized to the situation when the domain of the mappings is Δ, is the following:

Theorem BLMM. [1, page 310] *Suppose $|\mu_f(z)| \equiv k$. Then f is uniquely extremal for its boundary values on $\partial\Delta$ if and only if for every $E \subset \Delta$ with measure $|E| > 0$,*

$$inf \left\{ \frac{1}{\iint_E |\varphi(z)|\, dx\, dy} \left(k - \Re \iint_\Delta \mu_f(z)\varphi(z)\, dx\, dy \right) : \varphi \in L^1_a(\Delta)\,, \|\varphi\| = 1 \right\} = 0\,.$$

Since we want to give as constructive as possible a proof of the "only-if" part, our object is to derive the following portion of the BLMM Theorem in as constructive manner as we can:

Theorem. *Suppose f^μ is extremal for its boundary values* on $\partial\Delta$, *$|\mu(z)| \equiv k$, and suppose for some $E \subset \Delta$, with $|E| > 0$,*

$$inf \left\{ \frac{1}{\iint_E |\varphi(z)|\, dx\, dy} \left(k - \Re \iint_\Delta \mu_f(z)\varphi(z)\, dx\, dy \right) : \varphi \in L^1_a(\Delta)\,, \|\varphi\| = 1 \right\} \geq \gamma > 0\,. \tag{1.1}$$

Then there exists $\nu \in L^\infty(\Delta)$, with $\|\nu\|_\infty \leq k$, $\nu \neq \mu$, such that

$$f^\mu\Big|_{\partial\Delta} = f^\nu\Big|_{\partial\Delta}\,. \tag{1.2}$$

To a major extent we proceed by merely reworking parts of the analysis found in [4] and [1]. The author wishes to express his appreciation for the opportunity of numerous discussions with V. Marković in this connection.

2. Construction of f^ν.

Following [1, page 303], we start by choosing $r > 0$, and defining a modified version [1] α of μ, in terms of the set E of (1.1) as

$$\alpha(z) = \begin{cases} \frac{\mu(z)}{1+r}, & z \in \Delta \setminus E \\ \mu(z), & z \in E \end{cases}$$

Lemma. *Suppose f^μ is extremal for its boundary values on $\partial\Delta$, $\|\mu\|_\infty = k$, and suppose that (1.1) holds for some $E \subset \Delta$, with $|E| > 0$. Then*

$$K_0[f^\alpha] = \frac{1+r+k}{1+r-k}, \quad \left(0 < r \leq \frac{1-k}{1+k}\frac{\gamma}{k}\right). \tag{2.1}$$

Proof. Let $z_m = e^{2\pi i m/n}$, $w_m = f^\alpha(z_m)$, $m = 1, 2, \ldots n$. The closure of Δ with the distinguished sets of points, $\{z_m\}$, $\{w_m\}$, respectively, defines the respective n-gons, Δ_n and Δ_n^*. Among the class of quasiconformal mappings of Δ_n onto Δ_n^* for which $z_m \mapsto w_m$, $m = 1, 2, \ldots n$, there is [10] a unique quasiconformal mapping g_n with minimal $K[g_n]$. If we write the minimal value of $K[g_n]$ as $K_n = (1+k_n)/(1-k_n)$, then μ_{g_n} is of Teichmüller type,

$$\mu_{g_n}(z) = k_n \frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}.$$

If $n \geq 4$, and if $\varphi_n(z)$ is normalized to have $\|\varphi_n\| = 1$, $\varphi_n(z)$ is a uniquely determined rational function belonging to $L_a^1(\Delta)$. It is easy to show that $lim_{n\to\infty} K_n = K_0[f^\alpha]$. Furthermore, the sequence, $\{g_n\}$, will contain a subsequence converging uniformly in the closure of Δ to an extremal mapping f^η for the boundary values $f^\alpha|_{\partial\Delta}$. Moreover [8, Theorem 7], and for our purposes this is the most significant property of the constructed sequence, $\{g_n\}$,

$$K_n \leq \iint_\Delta |\varphi_n(z)| \frac{\left|1 + \alpha(z)\varphi_n(z)/|\varphi_n(z)|\right|^2}{1 - |\alpha(z)|^2}\, dx\, dy. \tag{2.2}$$

Setting $\lambda = \mu/(1+r)$, $t = k/(1+r)$, we can write the right side of (2.2) as $A + B$, where

$$A = \iint_\Delta |\varphi_n| \frac{\left|1+\lambda\varphi_n/|\varphi_n|\right|^2}{1-|\lambda|^2}\, dx\, dy, \; B = \iint_E |\varphi_n| \left[\frac{\left|1+\mu\varphi_n/|\varphi_n|\right|^2}{1-|\mu|^2} - \frac{\left|1+\lambda\varphi_n/|\varphi_n|\right|^2}{1-|\lambda|^2}\right] dx\, dy.$$

For A we have

$$\begin{aligned} A &\leq \frac{1}{1-t^2}\left\{(1+t^2) + \frac{2}{1+r}\Re \iint_\Delta \mu\varphi_n\, dx\, dy\right\} \\ &\leq \frac{1}{1-t^2}\left\{(1+t^2) + \frac{2}{1+r}\left[k - \gamma \iint_E |\varphi_n|\, dx\, dy\right]\right\} \\ &= \frac{1}{1-t^2}\left\{1 + t^2 + 2t - \frac{2\gamma}{1+r}\iint_E |\varphi_n|\, dx\, dy\right\} \\ &= \frac{1+t}{1-t} - \frac{2(1+r)\gamma}{(1+r)^2 - k^2}\iint_E |\varphi_n|\, dx\, dy. \end{aligned}$$

[1] An analogous modification procedure occurs in [4, pp. 109-110] in a related context. There the procedure is referred to as "lifting", and credited to Marković.

Furthermore,

$$B = 2(2r+r^2)\iint_E \frac{|\mu|^2|\varphi_n|}{(1-|\mu|^2)[(1+r)^2-|\mu|^2]}\,dx\,dy$$
$$+2r\Re\iint_E \frac{(1+r+|\mu|^2)\mu\varphi_n}{(1-|\mu|^2)[(1+r)^2-|\mu|^2]}\,dx\,dy$$
$$\le \frac{2rk(1+r+k)}{(1-k)[(1+r)^2-k^2]}\iint_E |\varphi_n|\,dx\,dy\,.$$

Hence,

$$A+B \le \frac{1+t}{1-t} + \frac{2}{(1+r)^2-k^2}\left\{rk\frac{1+r+k}{1-k} - (1+r)\gamma\right\}\iint_E |\varphi_n|\,dx\,dy\,.$$

When $0 < r \le (1-k)\gamma/[(1+k)k]$, the quantity in the brackets is negative. We therefore conclude that

$$K_0[f^\alpha] = K[f^\eta] \le \frac{1+r+k}{1+r-k}\,. \tag{2.3}$$

Since f^η has the same boundary values as f^α, it follows that $f^\mu \circ (f^\alpha)^{-1} \circ f^\eta$ has the same boundary values as f^μ. Since f^μ is extremal by hypothesis, it therefore follows that

$$\frac{1+k}{1-k} = K[f^\mu] \le K[f^\mu \circ (f^\alpha)^{-1} \circ f^\eta] \le K[h]K[f^\eta]\,, \tag{2.4}$$

where $h = f^\mu \circ (f^\alpha)^{-1}$. Since

$$|\mu_h(f^\alpha(z))| = \left|\frac{\mu(z)-a(z)}{1-\overline{a(z)}\mu(z)}\right| = \begin{cases} \frac{r|\mu(z)|}{1+r-|\mu(z)|^2}, & z \in \Delta\backslash E \\ 0, & z \in E \end{cases}$$

we have

$$|\mu_h(f^\alpha(z))| \le \frac{rk}{1+r-k^2}\,,\quad z \in \Delta\,.$$

Thus,

$$K[h] \le \frac{1+k}{1-k}\,\frac{1+r-k}{1+r+k}\,. \tag{2.5}$$

Substituting (2.5) into (2.4), we conclude that

$$K[f^\eta] \ge \frac{1+r+k}{1+r-k}\,.$$

Together with (2.3), the lemma follows. //

Proof. *of the the theorem.* We let

$$f^\nu = f^\mu \circ (f^\alpha)^{-1} \circ f^\eta\,,$$

thus insuring that (1.2) holds. Since $|\mu(z)| \equiv k$, it is clear, by the definition of α and the Lemma, that $\eta \ne \alpha$. Hence, $f^\nu \ne f^\mu$. On the other hand, since the inequality signs in (2.4) are replaced by equal signs, $K[f^\nu] = K[f^\mu]$. Therefore, f^ν is also an extremal mapping.

3. An example.

If one studies the foregoing construction, one sees that it is at least in the set E of (1.1) where the new extremal mapping differs from the given extremal mapping. In the preceding, Δ can be replaced by any simply-connected region Ω of hyperbolic type, the various expressions occuring in the statements of Theorem BLMM and the Lemma being invariant under appropriately performed transfers under conformal mappings. Boundary values are understood as being taken on at prime ends of Ω.

To illustrate the theorem, we may use Strebel's example of the affine stretch,

$$f_0(z) = K_0 x + iy\,,\ \ z = x + iy \in \Omega\,,\ (K_0 > 1)\,,$$

of the "chimney" region,

$$\Omega = \{z : \Im z < 0 \; or \; |\Re z| < 1\}\,,$$

referred to in Section 1. We start with the known fact that f_0 is extremal for its boundary values on $\partial\Omega$, and we will now show that if we take $E = \{\Im z < 0\}$ and $S = \Omega \setminus E = \{z : |\Re z| < 1,\ \Im z \geq 0\}$, then

$$inf \left\{ \frac{1}{\iint_E |\varphi(z)|\, dx\, dy} \left(k - \Re \iint_\Omega \mu_f(z)\varphi(z)\, dx\, dy \right) : \varphi \in L_a^1(\Omega)\,, \|\varphi\| = 1 \right\} \geq \gamma \quad (3.1)$$

is satisfied for some $\gamma > 0$, implying that f_0 is not the sole extremal mapping for its boundary values [2].

Since

$$\mu_{f_0}(z) \equiv k_0 = \frac{K_0 - 1}{K_0 + 1}\,,$$

we focus attention on

$$\rho = \frac{1 - \Re \iint_\Omega \varphi\, dx\, dy}{\iint_E |\varphi|\, dx\, dy}\,,\ \ \varphi \in L_a^1(\Omega)\,,\ \iint_\Omega |\varphi|\, dx\, dy = 1\,.$$

Since E is a half-plane, and $\varphi \in L_a^1(\Omega) \subset L_a^1(E)$, Cauchy's Theorem implies that

$$\iint_E \varphi\, dx\, dy = 0\,.$$

Therefore,

$$\rho = \frac{1 - \Re \iint_S \varphi\, dx\, dy}{\iint_E |\varphi|\, dx\, dy} \geq \frac{1 - \iint_S |\varphi|\, dx\, dy}{\iint_E |\varphi|\, dx\, dy} = \frac{\iint_E |\varphi|\, dx\, dy}{\iint_E |\varphi|\, dx\, dy} = 1\,.$$

So, (3.1) holds with $\gamma = k_0$. Whenever $0 < r \leq 1/K_0$, the construction procedure of Section 2 is guaranteed to yield an extremal mapping with the same boundary values as f_0 that differs from f_0. Actually, it turns out that for this example no upper bound on r need be set.

Following the procedure of Section 2, setting $T_0 = (1 + r + k_0)/(1 + r - k_0)$, we obtain

$$f^\alpha(z) = \begin{cases} T_0 x + iy, & y > 0 \\ T_0 x + (T_0/K_0) y\,, & y \leq 0 \end{cases}.$$

For f^η the simplest choice is

$$f^\eta(z) = T_0 x + iy\,,\ \ z = x + iy \in \Omega\,.$$

(Note that $K_0[f^\alpha] = K[f^\eta] = T_0$, precisely as predicted by the Lemma of Section 2.) With this choice of f^η, the extremal mapping with the same boundary values as f_0 that results is

$$f_1(z) = (f_0 \circ (f^\alpha)^{-1} \circ f^\eta)(z) = \begin{cases} K_0 x + iy\,, & z \in S \\ K_0 x + i(K_0/T_0) y\,, & \Im z \leq 0 \end{cases}.$$

[2] Since this is precisely what Strebel already showed in [9], we are of course doing this solely for the purpose of illustrating the particular construction procedure discussed above.

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