LIBERTAS MATHEMATICA, VOL XX (2000)

ON THE SOLVABILITY OF COMPLEMENTARITY PROBLEMS BY LERAY-SCHAUDER TYPE ALTERNATIVES

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1. Introduction

The Complementarity Theory is a relatively new domain of applied mathematics, with deep relations with several chapters of fundamental mathematics as for example: the fixed point theory, the theory of variational inequalities, the topological degree, the functional analysis, the theory of cones in topological vector spaces etc., [4], [7], [11]-[13], [25].

The main goal of Complementarity Theory is the study of several kinds of complementarity problems. Each complementarity problem is a mathematical model for several kinds of practical problems from economics, optimization, game theory, engineering and mechanics among others, [4], [7], [10], [11]-[13], [22], [25], [27].

Recently in [18], [5] we introduced a new topological method in the study of solvability of complementarity problems in Hilbert spaces. Our method is based on the concept of "exceptional family of elements" (shortly denoted by *EFE*). This notion is based on the topological degree and more general on the concept of "zero-epi mapping", [13], [14] and because this aspect, it is different of the notion of "exceptional sequence of elements" introduced in [27], which is strongly dependent of the Euclidean structure in $\mathbf{R}^n$ and of the ordering defined by $\mathbf{R}^n_+$.

By applying the notion of (*EFE*) several existence results for the nonlinear complementarity problem are proved in [5], [6], [14]-[22], [28]-[31].

In our papers, [15], [17], [21] it is shown that in the method of (*EFE*) we can replace the topological degree by Leray-Schauder type alternatives. By this way the proofs are substantially more simple.

Now, in this paper we will present two new applications of Leray-Schauder type alternatives to the study of complementarity problems.

In the first application we will obtain two new existence theorems for the nonlinear complementarity problem, while in the second application we will obtain a generalization to α-condensing fields of the main existence result proved in our paper [14] for k-set fields.

2. Preliminaries

Let $(H, < \cdot, \cdot >)$ be a Hilbert space and $K \subset H$ a closed pointed convex cone, i.e., K is a non-empty closed set satisfying the following properties: (1) $K + K \subseteq K$; (2) $\lambda K \subseteq K$, for all $\lambda \in \mathbf{R}_+$ and (3) $K \cap (-K) = \{0\}$.

The dual of K is, by definition, $K^* = \{y \in K | < x, y > \geq 0 \text{ for all } x \in K\}$. It is well known that K^* is a closed convex cone. If $K \subset H$ is a closed pointed convex cone, then the *projection* onto K, denoted by P_K, is well defined for every $x \in H$, i.e., for every $x \in H$, $P_K(x)$ is the unique element in K such that $\|x - P_K(x)\| = \min\limits_{y \in K} \|x - y\|$.

A classical result says that the projection operator P_K is characterized by the following properties. For every $x \in H$, $P_K(x)$ is the unique element in K satisfying the following conditions:

(i) $\langle P_K(x) - x, y \rangle \geq 0$, for all $y \in K$,

(ii) $\langle P_K(x) - x, P_K(x) \rangle = 0$.

3. Leray-Schauder type alternatives

One of the most important theorem of Nonlinear Functional Analysis is the *Leray-Schauder Alternative*, which was proved initially in Banach spaces by the *topological degree* in 1934 [24].

Now, there exist several kings of Leray-Schauder Alternative proved without topological degree [2], [3], [9], [26].

It is also well known that the classical Leray-Schauder Alternative has many applications to ordinary differential equations and to partial differential equations.

Our applications of Leray-Schauder type alternatives to the study of complementarity problems represent a new direction of applications of this classical result.

In this paper we will apply the following Leray-Schauder type alternatives.

Let $(H, < \cdot, \cdot >)$ be a Hilbert space, $X \subset H$ a non-empty subset and $f : X \rightarrow H$ a mapping. The mapping f is said to be *compact* if $f(X)$ us relatively compact. We say that f is *completely continuous* if f is continuous and for any bounded set $B \subset H$, $f(B)$ is relatively compact. We will use also the following classical notion. We say that a mapping $f : H \rightarrow H$ is a *completely continuos field*, if f has a representation of the form $f(x) = x - T(x)$, for every $x \in H$, where $T : H \rightarrow H$ is a completely continuous mapping. Given a non-empty subset X of H, we will denote by $\overline{X}$ the closure of X and by ∂X the boundary of X.

Theorem 1 [Leray-Schauder Alternative] *Let $C \subset H$ be a convex set, U a subset open in C and such that $0 \in U$. Then each continuous compact mapping $f : \overline{U} \rightarrow C$ has at least one of the following two properties:*

(1) *f has a fixed point,*

(2) *there is $(x_*, \lambda_*) \in \partial U \cdot]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof. A proof of this result based on transversality theory is given in [9]. ■

Let $\mathbf{K} \subset H$ be a closed pointed convex cone. For any $r > 0$, $(r \in \mathbf{R})$ we denote by $\mathbf{K}_r = \{x \in \mathbf{K} | \|x\| \leq r\}$. If $f : \mathbf{K}_r \rightarrow E$ is a mapping, we recall that f is α-*condensing* (where α is the Kuratowski measure of noncompactness [1]), if f is continuous and $\alpha(f(B)) < \alpha(B)$, for all $B \subset \mathbf{K}_r$ such that $\alpha(B) > 0$. For more information about α-condensing mappings the reader is referred to [1].

The next Leray-Schauder type alternative is based on the following fixed point theorem.

Theorem 2 [Deimling] *Let $(E, \|\cdot\|)$ be a Banach space, $K \subset E$ a closed pointed convex cone and $f : K_r \to E$ an α-condensing mapping. If the following assumptions are satisfied:*

(1) $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ *and* $x^*(x) = 0$, *then*

(2) $f(x) \neq \lambda x$ *on* $\|x\| = r$ *for all* $\lambda > 1$,

then f has a fixed point (in K_r).

Proof. A proof of this result is in [8]. ■

A consequence of *Theorem* 2 is the following Leray-Schauder type alternative.

Theorem 3 *Let $(H, < \cdot, \cdot >)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $h : K \to H$ a mapping such that $h(x) = x - T(x)$, for all $x \in H$, where $T : H \to H$ is an α-condensing mapping.*

Then, for any $r > 0$, for the mapping $f(x) = P_K[x - h(x)]$, at least one of the following two situations is satisfied:

(1) *f has a fixed point in K_r,*

(2) *there exist x_*, with $\|x_*\| = r$ and $\lambda_* \in]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof. Since $\alpha(P_K[T(B)]) \leq \alpha[T(B)] < \alpha(B)$, for all $B \subset K_r$ with $\alpha(B) > 0$, we have that f is α-condensing and all the assumptions of *Theorem* 2 are satisfied. ■

4. Complementarity problems

Let $(H, < \cdot, \cdot >)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ an arbitrary mapping. The Nonlinear Complementarity Problem defined by the mapping f and the cone K is:

$$NCP(f, K) : \begin{cases} \text{find } x_* \in K \text{ such that} \\ \quad f(x_*) \in K^* \text{ and} \\ \quad \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

The problem $NCP(f, K)$ has many applications and generally, it is related to equilibrium problems in the physical or economical sense [4], [7], [11]-[13], [22], [25], [27]. If the mapping f is an affine mapping, i.e., $f(x) = A(x) + b$, where $A : H \to H$ is a continuous linear mapping and b is an arbitrary element in H, we have *the linear complementarity problem*, denoted by $LCP(A, b\ K)$. The problem $LCP(A, b\ K)$ has been very much studied in the Euclidean space $(\mathbf{R}^n, < \cdot, \cdot >)$.

In the Complementarity Theory, the study of the solvability of the problem $NCP(f, K)$ is the first important problem, because the solvability of this problem is absolutely not evident, [4], [7], [11]-[13], [25].

5. The solvability of complementarity problem for ρ-quasi-bounded fields

Let $(H, <\cdot,\cdot>)$ be a Hilbert space, and $T : H \to H$ a mapping. We say that T is a quasi-bounded mapping if

$$|T|_{qb} := \inf_{r>0} \sup_{\|x\|\geq r} \frac{\|T(x)\|}{\|x\|} < +\infty$$

The real number $|T|_{qb}$ is called the quasi-norm of T. The notion of *quasi-bounded* mapping was introduced by A. Granas (see [9] and references).

Definition 1 *We say that a mapping $f : H \to H$ is a ρ-quasi-bounded field if f has a representation of the form $f(x) = \rho x - T(x)$ for all $x \in H$, where $\rho \in R_+\backslash\{0\}$ and $T : H \to H$ is a completely continuous quasi-bounded mapping.*

Remark If, for a mapping $T : H \to H$, we have that $\beta = \lim\limits_{\|x\|\to\infty} \dfrac{\|T(x)\|}{\|x\|} < +\infty$, then we have that $|T|_{qb} = \beta$.

Theorem 4 *Let $(H, <\cdot,\cdot>)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a ρ-quasi-bounded field with the representation $f(x) = \rho x - T(x)$, for all $x \in H$. If $|T|_{qb} < \rho$, then the problem $NCP(f, K)$ has a solution.*

Proof. Consider the mapping $\Phi(x) =$, for all $x \in H$ and observe that the problem $NCP(f, K)$ has a solution, if and only if, the mapping Φ has a fixed point.

We have that

$$|\Phi|_{qb} = \inf_{r>0} \sup_{\|x\|\geq r} \frac{\|\Phi(x)\|}{\|x\|} = \inf_{r>0} \sup_{\|x\|\geq r} \frac{\left\|P_K\left(\frac{1}{\rho}T(x)\right)\right\|}{\|x\|} < \inf_{r>0} \sup_{\|x\|\geq r} \frac{\frac{1}{\rho}\|T(x)\|}{\|x\|} = \frac{1}{\rho}|T|_{qb} < 1.$$

Hence, there exists $\rho > 0$ such that

$$\frac{\|\Phi(x)\|}{\|x\|} < 1, \text{ for all } x \text{ with } \|x\| > r. \tag{1}$$

We take $C = H$ and $U = \{x \in H \,|\, \|x\| < r\}$. We obtain that, there is no $x_* \in U$ and $0 < \lambda_* < 1$ such that $x_* = \lambda_* \Phi(x_*)$. Indeed, if a such x_* and λ_* exist we have

$$\|x_*\| = \lambda_* \|\Phi(x_*)\| < \|\Phi(x_*)\|$$

which is a contradiction of (1).

The assumptions of *Theorem 1* are satisfied for Φ. Therefore, by *Theorem 1* the mapping Φ has a fixed point, which implies that the problem $NCP(f, K)$ has a solution. ■

The following result is a consequence of *Theorem 4* and it is an existence result for nonlinear complementarity problems depending of a real parameter.

Theorem 5 *Let $(H, <\cdot,\cdot>)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. Consider the mapping $f_\varepsilon(x) = \rho x - T_1(x) - \varepsilon T_2(x)$, for all $x \in H$, where ρ is a positive real number and ε is a positive real parameter, $T_1 : H \to H$ is a completely continuous linear operator such that $\|T_1\| < \rho$ and $T_2 : H \to H$ is a completely continuous quasi-bounded mapping.*

Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ the problem $NCP(f_\varepsilon, K)$ has a solution $x_*(\varepsilon)$.

Proof. The operator $T_1 - \varepsilon T_2$ is completely continuos for every $\varepsilon > 0$. Since $||T_1|| < \rho$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, $\varepsilon |T_2|_{qb} < \rho - ||T_1||$. Since, $|T_1 + \varepsilon T_2|_{qb} \leq ||T_1|| + \varepsilon |T_2|_{qb} < \rho$, for all $\varepsilon \in]0, \varepsilon_0[$, then by *Theorem 4* we have that for each $\varepsilon \in]0, \varepsilon_0[$, the problem $NCP(f_\varepsilon, K)$ has a solution $x_*(\varepsilon)$. ■

Remark The operator f_ε is a generalization of the Von Karman operator $f(x) = x - \lambda L(x) + T(x)$ used in the study of the post critical equilibrium of thin elastic plates. In this case the mathematical model is the problem $NCP(f, K)$ [12], [13].

6. Exceptional family of elements for α-condensing fields and complementarity problems

Let $(H, < \cdot, \cdot >)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. Consider the problem $NCP(f, K)$ defined by f and K.

Definition 2 *We say that a family of elements $\{x_r\}_{r>0}$ is an exceptional family of elements for a mapping $f : H \to H$, with the respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:*
(1) $u_r \in K^*$,
(2) $\langle u_r, x_r \rangle = 0$,
(3) $||xr|| \to +\infty$ *as* $r \to +\infty$.

We have the following alternative theorem for complementarity problems.

Theorem 6 *Let $(H, < \cdot, \cdot >)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. If f is an α-condensing field, i.e., $f(x) = x - T(x)$, where $f : H \to H$ is an α-condensing mapping, then there exists either a solution to the problem $NCP(f, K)$, or an exceptional family of elements for f with respect to K.*

Proof. We consider the mapping

$$\Phi(x) = P_K[x - f(x)] = P_K[T(x)], \text{ for all } x \in H.$$

From the complementarity theory, we know that the problem $NCP(f, K)$ has a solution, if and only if, the mapping Φ has a fixed point (in K), [11]-[13].

Therefore, if the mapping Φ has a fixed point, then the problem $NCP(f, K)$ has a solution. Suppose that the problem $NCP(f, K)$ is without solution. Obviously, in this case the mapping Φ is fixed-point free.

We observe that the assumptions of *Theorem 3* are satisfied. Then for any $r > 0$ there exist, x_r with $||x_r|| = r$ and $\lambda_r \in]0, 1[$ such that $x_r = \lambda_r P_k[T(x_r)]$. We have

$$\frac{1}{\lambda_r} x_r = P_K[x_r - f(x_r)]. \tag{2}$$

Applying the properties of P_K, we obtain

$$\begin{cases} \left\langle \dfrac{1}{\lambda_r} x_r - (x_r - f(x_r)), y \right\rangle \geq 0, \text{ for all } y \in K \\ \text{and} \\ \left\langle \frac{1}{\lambda_r} x_r - (x_r - f(x_r)), \dfrac{1}{\lambda_r} x_r \right\rangle = 0 \end{cases}$$

which implies

$$\left\{\begin{array}{l} \left\langle \left(\dfrac{1}{\lambda_r}-1\right)x_r+f(x_r),y\right\rangle \geq 0,\ \textit{for all } y\in K \\ \textit{and} \\ \left\langle \left(\dfrac{1}{\lambda_r}-1\right)x_r+f(x_r),\dfrac{1}{\lambda_r}x_r\right\rangle = 0. \end{array}\right. \tag{3}$$

If in (3) we put $\mu_r = \dfrac{1}{\lambda_r} - 1$, it follows that $\mu_r x_r + f(x_r) \in K^*$, $\langle \mu_r x_r + f(x_r), x_r\rangle = 0$ and since for any $r > 0$, $||x_r|| = r$, we have (because $x_r \in K$) that $\{x_r\}_{r\geq 0}$ is an exceptional family of elements for f. ■

Corollary *Let* $(H, < \cdot, \cdot >)$ *be a Hilbert space and* $K \subset H$ *a closed pointed convex cone. If* $f : H \to H$ *is an* α*-condensing field without exceptional family of elements, with respect to* K*, then the problem* $NCP(f, K)$ *has a solution.*

Considering the *Corollary* of *Theorem 6* we deduce that an important fact is to know, what functions are without exceptional families of elements with respect to a given convex cone.

The reader can find in our papers, [14], [17]-[20] and in our book [13] that several classes of functions are without exceptional family of elements. It is known [18] that any coercive function is without exceptional family of elements, but there exist functions without family of elements, which are not coercive.

Now, we recall a condition, introduced in [14] and [19], which implies the non-existence of an exceptional family of elements.

We say that a mapping $f : H \to H$ *satisfies condition* (θ) *with respect to a convex cone* $K \subset H$ *if there exists* $\rho > 0$ *such that for each* $x \in K$ *with* $||x|| > \rho$*, there exists* $y \in K$ *with* $||y|| < ||x||$ *such that* $\langle x - y, f(x)\rangle \geq 0$.

We proved in [14]-[17] that, several classes of mappings used in complementarity theory satisfy condition (θ) and we proved also in [14] that any mapping satisfying condition (θ) is without exceptional family of elements.

In conclusion we have the following result.

Theorem 7 *Let* $(H, < \cdot, \cdot >)$ *be a Hilbert space and* $K \subset H$ *a closed pointed convex cone. If* $f : H \to H$ *is an* α*-condensing field satisfying condition* (θ)*, then the problem* $NCP(f, K)$ *has a solution.*

7. Comments

It is interesting to discover new classes of mappings without exceptional families of elements. By this way we can obtain new existence theorems for nonlinear comple-mentarity problems. Because, the method of exceptional family of elements is now used in the study of solvability of variational inequalities, [21], [28-[31], it is interesting to know if we can extend to a-condensing fields the results obtained in [21], [28]-[31].

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