LIBERTAS MATHEMATICA, VOL. XIX (1999)

# SOME DEVELOPMENTS OF STRONGLY NONLINEAR POTENTIAL THEORY

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**Abstract.** *In this article we establish some relations between potentials and maximal functions in Orlicz spaces. We give a definition of quasicontinuity and obtain a description of quasicontinuous representative in some potential spaces. We also give a result on smooth truncation of potentials in Orlicz-Sobolev spaces and compare some capacities. As a consequence: a compact is removable in Orlicz space for an elliptic linear operator of order m with constant coefficients if and only if its Bessel capacity of order m is null.*

## 1. INTRODUCTION

The Nonlinear Potential Theory has known a great development since the works of N.G. Meyers [19] and those of V.P. Havin and V.G. Maz'ya [11]. The book of D.R. Adams and L.I. Hedberg [2] shows the importance of this development. We have introduced in [5,6,7] a Theory of Potential in Orlicz spaces, called Strongly Nonlinear Potential Theory, developed later in some directions. In this paper we continue this development for other components of this Theory.

We establish in §2 some relations between maximal functions and Riesz and Bessel Potentials with the aid of Orlicz norm. The obtained results generalize among other things, the celebrated theorem of B. Muckenhoupt and R.L. Wheeden for $L^p$ Lebesgue spaces without weight.

The notion of quasicontinuity in Orlicz spaces is introduced in §3. The potential is quasicontinuous when the N-function defining the Orlicz space verifies the $\Delta_2$ condition. We show also uniqueness of the quasicontinuous representative for Bessel kernels and for any N-function. The same result remains valid for any kernel but for reflexive Orlicz spaces.

For reflexive Orlicz spaces, we show in §4 that composition of Bessel potential with a smooth operator, is a potential. This is an extension of a well known Theorem of V.G. Maz'ya which is a substitute of the fact that the Sobolev spaces $W^{m,p}(m \neq 1)$ are not closed under contractions. An immediate consequence is the equivalence of capacities $N_{m,A}$ and $B_{m,A}$ (see Theorem 2, §6). Note that in the case of $L^p$ Lebesgue spaces, this two capacities are equivalent even if $m$ is not integer. (See [3]). The correspondent case for Orlicz spaces remains open.

We show in §5 that in reflexive Orlicz spaces, a compact set $K$ is removable for an elliptic linear operator of order $m$, with constant coefficients, if and only if its Bessel capacity is null (i.e. $B_{m,A}(K) = 0$). This is the first relation between the Strongly Nonlinear Potential Theory, and Partial Differential Equations. We hope that other relations will be established as in the case of the two Theories Linear and Nonlinear.

## 2. PRELIMINARIES

### 2.1 Orlicz spaces

We recall some principal definitions and results about Orlicz spaces. The classical references are [4, 15, 16, 17].

Let $A : R \to R^+$ be an *N-function*, i.e. $A$ is continuous, convex, with $A(t) > 0$ for $t > 0$, $\lim\limits_{t\to 0} \dfrac{A(t)}{t} = 0$, $\lim\limits_{t\to +\infty} \dfrac{A(t)}{t} = +\infty$ and $A$ is even.

Equivalently, $A$ has the representation: $A(t) = \displaystyle\int_0^{|t|} a(x)dx$, where $a : R^+ \to R^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $\lim\limits_{t\to +\infty} a(t) = +\infty$.

The N-function $A^*$ *conjugate* to $A$ is defined by $A^*(t) = \displaystyle\int_0^{|t|} a^*(x)dx$, where $a^*$ is given by $a^*(s) = \sup\{t : a(t) \leq s\}$.

Let $A$ be an N-function and let $\Omega$ be an open set in $R^N$. We note $\mathcal{L}_A(\Omega)$ the set, called an *Orlicz class*, of measurable functions $f$ on $\Omega$, such that

$$\rho(f, A, \Omega) = \textstyle\int_\Omega A(f(x))dx < \infty.$$

Let $A$ and $A^*$ be two conjugate N-functions and let $f$ be a measurable function defined almost everywhere in $\Omega$. The *Orlicz norm of* $f$, $||f||_{A,\Omega}$ or $||f||_A$ if there is no confusion, is defined by

$$||f||_A = \sup\left\{\int_\Omega |f(x)g(x)|\,dx : g \in \mathcal{L}_{A^*}(\Omega) \text{ and } \rho(f, A^*, \Omega) \leq 1\right\}.$$

The set $\mathbf{L}_A(\Omega)$ of measurable functions $f$, such that $||f||_A < \infty$ is called an *Orlicz space*. When $\Omega = R^N$, we set $\mathbf{L}_A$ in place of $\mathbf{L}_A(R^N)$.

The *Luxemburg norm* $|||f|||_{A,\Omega}$ or $|||f|||_A$ if there is no confusion, is defined in $\mathbf{L}_A(\Omega)$ by

$$|||f|||_A = \inf\left\{r > 0 : \int_\Omega A(\tfrac{f(x)}{r})dx \leq 1\right\}.$$

Let $A$ be an N-function. We say that $A$ *verifies the* $\Delta_2$ *condition* if there exists a constant $C > 0$ such that $A(2t) \leq CA(t)$ for all $t \geq 0$.

Recall that $A$ verifies the $\Delta_2$ condition if and only if $\mathcal{L}_A = L_A$. Moreover $L_A$ is reflexive if and only if $A$ and $A^*$ verify the $\Delta_2$ condition.

Let $m$ be a positive integer. The Orlicz-Sobolev space $W^m L_A(\Omega)$ is the space of real functions $f$, such that $f$ and its distributional derivatives up to the order $m$, are in $L_A(\Omega)$. $W^m L_A(\Omega)$ is a Banach space equipped with the norm:

$$|||f|||_{m,A} = \sum_{|i|\leq m} |||D^i f|||_A,\ f \in W^m L_A(\Omega).$$

Let $W^{-m} L_A(\Omega)$ denote the space of distributions on $\Omega$, which can be written as sums of derivatives up to the order $m$ of functions in $L_{A^*}(\Omega)$. It is a Banach space under the usual quotient norm.

Recall that if $A$ and $A^*$ satisfy the $\Delta_2$ condition, the dual of $W^m L_A(R^N)$ coincides with $W^{-m} L_A(R^N)$.

## 2.2 Capacity and Potential in Orlicz Spaces

We shall need some definitions and results concerning capacities and potentials in Orlicz spaces. For more details, see [5, 6, 7].

**Definition 1** *Let $\Gamma$ be a $\sigma$-additive class of sets, which contains compact sets in $R^N$. Let $C$ be a positive function defined in $\Gamma$.*

*A) $C$ is called capacity if it satisfies the following axioms*

*(i) $C(\emptyset) = 0$.*

*(ii) If $X$ and $Y$ are in $\Gamma$ and $X \subset Y$, then $C(X) \leq C(Y)$.*

*(iii) If $X_i$, $i = 1, 2, \ldots$ are in $\Gamma$, then $C(\bigcup_{i\geq 1} X_i) \leq \sum_{i\geq 1} C(X_i)$.*

*B) $C$ is called an outer capacity if for every $X \in \Gamma$,*

$$C(X) = \inf\{C(O) : O \text{ open}, X \subset O\}.$$

*C) $C$ is called an inner capacity if for every $X \in \Gamma$,*

$$C(X) = \sup\{C(K) : K \text{ compact}, K \subset X\}.$$

Let $k$ be a positive and measurable function in $R^N$, called a kernel, and let $A$ be an N-function. For $X \subset R^N$, we define

$$C_{k,A}(X) = \inf\{A(|||f|||_A) : f \in \mathbf{L}_A^+ \text{ and } k * f \geq 1 \text{ on } X\}$$
$$C'_{k,A}(X) = \inf\{|||f|||_A : f \in \mathbf{L}_A^+ \text{ and } k * f \geq 1 \text{ on } X\}$$

where $k*\ f$ is the usual convolution. The sign $+$ deals with positive elements in the considered space. From [6] $C'_{k,A}$ is a capacity.

If a statement holds except on a set $X$ where $C_{k,A}(X) = 0$, then we say that the statement holds $C_{k,A}$-*quasieverywhere* (abbreviated $C_{k,A} - q.e$ or $(k, A) - q.e$ if there is no confusion).

Let $f$ and the elements of the sequence $(f_i)_i$ be valued real functions which are finite $C_{k,A}-q.e.$ We say that the sequence $(f_i)_i$ converges $C_{k,A}$ quasiuniformly to $f$ (abbreviated $f_i \rightarrow f\ C_{k,A}-q.u$) if

$$\forall \epsilon > 0, \exists X : C_{k,A}(X) < \epsilon, \text{ and } f_i \rightarrow f \text{ uniformly on } {}^cX.$$

We call a function $f$ in $\mathbf{L}_A^+$ such that $k * f \geq 1$ on $X$, a test function for $C'_{k,A}(X)$. Moreover, a test function, say $f$, for $C'_{k,A}(X)$ such that for $C'_{k,A}(X) = |||f|||_A$ is called a $C'_{k,A}-$capacitary distribution for $X$ and $k * f$ is called a $C'_{k,A}-$capacitary potential for $X$.

$M$ denotes the vector space of Radon measures. $M_1$ is the Banach space of measures equipped with the norm $\parallel \mu \parallel$=total variation of $\mu < \infty$.

$F$ will stand for the $\sigma-$field of sets which are $\mu-$measurable for all $\mu \in M_1^+$.

If $\mu \in M_1^+$, we say that $\mu$ is concentrated on $X$ if $\mu(y) = 0$ for all sets $Y$ which are $\mu-$measurable and such that $Y \subset {}^cX$.

Let $A$ and $A^*$ be two conjugate N-functions. For $X \in F$, we define

$$D_{k,A}(X) = \sup \{\parallel \mu \parallel : \mu \in M_1^+,\ \mu \text{ concentrated on } X \text{ and } \parallel k * \mu \parallel_{A^*} \leq 1\}$$

where $k * \mu$ is the convolution of $k$ and $\mu$ defined by $(k * \mu)(x) = \displaystyle\int k(x-y)d\mu(y)$.

A measure $\mu \in M_1^+$ such that $\mu$ is concentrated on $X$ and $\parallel k * \mu \parallel_{A^*} \leq 1$ is called a test measure for $D_{k,A}(X)$. If in addition $D_{k,A}(X) = \parallel \mu \parallel$, we say that $\mu$ is a $D_{k,A}-$capacitary distribution for $X$ and $k * \mu$ is called a $D_{k,A}-$capacitary potential for $X$.

For the properties of $C_{k,A}$, $C'_{k,A}$, and $D_{k,A}$, see [5, 6, 7].

For $m > 0$, the *Bessel kernel*, $G_m$, is most easily defined through its Fourier transform $F(G_m)$ as: $[F(G_m)](x) = (2\pi)^{-\frac{N}{2}}(1+ |x|^2)^{-\frac{m}{2}}$
where $[F(f)](x) = (2\pi)^{-\frac{N}{2}} \displaystyle\int f(y)e^{-ixy}dy$ for $f \in \mathbf{L}^1$.

$G_m$ is positive, in $\mathbf{L}^1$ and verifies the equality: $G_{r+s} = G_r * G_s$.

We put in the sequel $B_{m,A} = C_{G_m,A}$ and $B'_{m,A} = C'_{G_m,A}$.

Let $A$ be an N-function and $m > 0$. We define the space of potentials $\mathbf{L}_{m,A}$ as: $\mathbf{L}_{m,A} = \{\Psi = G_m * f : f \in \mathbf{L}_A\}$ equipped with the norm $|||\Psi|||_{m,A} = |||f|||_A$, when $\Psi = G_m * f$.

Note that when $m$ is integer and $A$ and $A^*$ satisfy the $\Delta_2$ condition, the spaces $W^m\mathbf{L}_A$ and $\mathbf{L}_{m,A}$ coincide and are of equivalent norms. For more details, see [10].

We note $I_m(x) = |x|^{m-N}$ the Riesz kernel. We have (see [8, 9, 23])

$$G_m(x) \sim I_m(x), when \mid x \mid \rightarrow 0, with 0 < m < N, \tag{1}$$

On the other hand, for every $c < 1$,

$$G_m(x) = O(e^{-c|x|}), when \mid x \mid \rightarrow \infty,\ with\ 0 < m. \tag{2}$$

Another inequality which serves in this paper is

$$G_m(x) \leq CG_m(x+y),\ \ |x| \geq 2,\ \ |y| \leq 1. \tag{3}$$

Pose $G_m(x) = G_m(r)$ if $r = |x|$. We have the following behaviors of $G'_m$ near 0 and infinity

$$G'_m(r) \sim -(N-m)G_{m-1}(r), m > 1, \ when \ r \to 0, \tag{4}$$

$$G'_m(r) \sim -b_m r^{\frac{m-N-1}{2}} e^{-r} \sim -c_m G_m(r), \ when \ r \to \infty \tag{5}$$

with $b_m = (4\pi)^{\frac{-m}{2}}[\Gamma(\frac{m}{2})]^{-1}$ and $c_m = 2^{\frac{N+m-3}{2}}\pi^{\frac{N-m-1}{2}}$.

## 3. MAXIMAL OPERATORS AND POTENTIALS

Let $f$ be a locally integrable function. The Hardy-Littlewood maximal function associated to $f$ is defined by $Mf(x) = M_0 f(x) = \sup\limits_{r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y)|\,dy$.

Here $|B(x,r)|$ is the Lebesgue measure of $B(x,r)$ on $R^N$.

The fractional maximal function associated to $f$ is defined for $0 < \alpha < N$, by $M_\alpha f(x) = \sup\limits_{r>0} |B(x,r)|^{\frac{\alpha-N}{N}} \int\limits_{B(x,r)} |f(y)|\,dy$.

And for $0 \le \alpha < N$, and $\delta > 0$, the inhomogeneous version of these functions is defined by $M_{\alpha,\delta} f(x) = \sup\limits_{\delta \ge r > 0} |B(x,r)|^{\frac{\alpha-N}{N}} \int\limits_{B(x,r)} |f(y)|\,dy$.

For $0 \le \alpha < N$, and $\delta > 0$, we define the modified Riesz kernel, $I_{\alpha,\delta}$ by

$$I_{\alpha,\delta}(x) = I_\alpha(x), \ if \ \ |x| < \delta,$$
$$I_{\alpha,\delta}(x) = 0, \ if \ \ |x| \ge \delta.$$

The Riesz potential $I_\alpha * \mu$, $0 < \alpha < N$, where $\mu$ is a positive measure, can be estimated below by the fractional maximal function associated to $\mu$. In fact, for every $r > 0$,

$$\int\limits_{R^N} |x-y|^{\alpha-N}\,d\mu(y) \ge \int\limits_{|x-y|\le r} |x-y|^{\alpha-N}\,d\mu(y) \ge \int\limits_{|x-y|\le r} d\mu(y).$$

The reverse inequality is false in general.

In the first part of the following theorem, we give a generalization to Orlicz spaces, of the classical theorem of B. Muckenhoupt and R.I. Wheeden [20], which establishes the opposite inequality in term of $L^p$ norms. Our method follows the one given by D.R. Adams and L.I. Hedberg in [2, Chapter 3], and by B.O. Turesson in [24, Chapter 3]. The second part generalizes a result by B. Jawerth, C. Pérez, G. Welland in [13, p.86].

**Theorem 1** *Let $A$ be an N-function satisfying the $\Delta_2$ condition, and let $0 < \alpha < N$. Then*

*1)There is a constant $C > 0$, such that for any positive measure $\mu$,*

*i)* $C^{-1} \int A(M_\alpha \mu)dx \le \int A(I_\alpha * \mu)dx \le C \int A(M_\alpha \mu)dx$

*ii)* $C^{-1}|||M_\alpha\mu|||_A \le |||I_\alpha * \mu|||_A \le C|||M_\alpha\mu|||_A$.

*2)If* $\delta$ *is a positive number, there are positive constants* $C_1$, $C_2$ *and* $C_3$ *such that for any positive measure* $\mu$,

$$1.\ \ |||M_{\alpha,\delta}\,\mu|||_A \le C_1|||I_{\alpha,\delta} * \mu|||_A \le C_2|||G_\alpha * \mu|||_A \le C_3|||M_{\alpha,\delta}\,\mu|||_A.$$

*Proof.* 1. We remark that for any positive measure $\mu$, we have $M_\alpha\mu \le CI_\alpha * \mu$.
Hence $C^{-1}\int A(M_\alpha\mu)dx \le \int A(I_\alpha * \mu)dx$,and $C^{-1}|||M_\alpha\mu|||_A \le |||I_\alpha * \mu|||_A$.
We must prove the opposite inequalities.
*i)* First, suppose $\mu$ with compact support.
Recall the so called "*good* $\lambda$ inequality" (see [2, Chapter 3, Th. 3.6.1]):
*There are* $c > 1$ and $b > 0$ such that for any $\lambda > 0$ and any $\epsilon$, $0 < \epsilon \le 1$,

$$|\{x : I_\alpha * \mu(x) > c\lambda\}| \le b\epsilon^{\frac{N}{N-\alpha}}\,|\{x : I_\alpha * \mu(x) > \lambda\}| + |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|.$$

This implies for any $L > 0$,

$$\int_0^L |\{x : I_\alpha * \mu(x) > c\lambda\}|\,A'(c\lambda)d\lambda \le b\epsilon^{\frac{N}{N-\alpha}}\int_0^L |\{x : I_\alpha * \mu(x) > \lambda\}|\,A'(c\lambda)d\lambda +$$
$$\int_0^L |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|\,A'(c\lambda)d\lambda.$$

We have

$$\int_0^L |\{x : I_\alpha * \mu(x) > c\lambda\}|\,A'(c\lambda)d\lambda = c^{-1}\int_0^{cL} |\{x : I_\alpha * \mu(x) > \lambda\}|\,A'(\lambda)d\lambda$$

and

$$\int_0^L |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|\,A'(c\lambda)d\lambda \le K_1\int_0^L |\{x : M_\alpha\mu(x) > \epsilon\lambda\}|\,A'(\lambda)d\lambda$$

$$\le \epsilon^{-1}K_1\int_0^{\epsilon L} |\{x : M_\alpha\mu(x) > \lambda\}|\,A'(\epsilon^{-1}\lambda)d\lambda \le \epsilon^{-1}K_1K_2\int_0^{\epsilon L} |\{x : M_\alpha\mu(x) > \lambda\}|\,A'(\lambda)d\lambda.$$

We obtain

$$c^{-1}\int_0^{cL} |\{x : I_\alpha * \mu(x) > \lambda\}|\,A'(\lambda)d\lambda \le Kb\epsilon^{\frac{N}{N-\alpha}}\int_0^L |\{x : I_\alpha * \mu(x) > \lambda\}|\,A'(\lambda)d\lambda +$$

$$\epsilon^{-1}K_1K_2\int\limits_0^{\epsilon L}|\{x : M_\alpha\mu(x) > \lambda\}|\, A'(\lambda)d\lambda.$$

Since the support of $\mu$ is compact, these integrals are finite.

We choose $\epsilon$ such that $Kb\epsilon^{\frac{N}{N-\alpha}} \le (2c)^{-1}$.

It follows that

$$(2c)^{-1}\int\limits_0^{cL}|\{x : I_\alpha * \mu(x) > \lambda\}|\, A'(\lambda)d\lambda \le \epsilon^{-1}K_1K_2\int\limits_0^{\epsilon L}|\{x : M_\alpha\mu(x) > \lambda\}|\, A'(\lambda)d\lambda.$$

When $L \to +\infty$, we get $\int A(I_\alpha * \mu)dx \le C\int A(M_\alpha\mu)dx$.

If the support of $\mu$ is not compact, we denote by $\mu_n$ the restriction of $\mu$ to the ball $B(0,n)$ for $n = 1, 2, \ldots$

Hence $\int A(I_\alpha * \mu_n)dx \le C\int A(M_\alpha\mu)dx, \forall n$, with $C$ independent of $n$. The monotone convergence theorem gives *i*).

*ii*) From *i*), we deduce that for any positive measure $\mu$,

$$\int A\left(\tfrac{I_\alpha * \mu}{|||M_\alpha\mu|||_A}\right)dx \le C\int A\left(\tfrac{M_\alpha\mu}{|||M_\alpha\mu|||_A}\right)dx \le C.$$

If $C \le 1$, for any positive measure $\mu$, we get $|||I_\alpha * \mu|||_A \le |||M_\alpha\mu|||_A$.

If $C > 1$, for any positive measure $\mu$, we get $\int A\left(\tfrac{I_\alpha * \mu}{C|||M_\alpha\mu|||_A}\right)dx \le 1$.

This implies $|||I_\alpha * \mu|||_A \le C|||M_\alpha\mu|||_A$.

2. As in 1., it is enough to consider positive measures with compact support. Without loose of generality we must take $\delta = 1$. We begin by showing the following

$$|||G_\alpha * \mu|||_A \le C|||I_{\alpha,1} * \mu|||_A + C|||M_{\alpha;1}\mu|||_A \tag{6}$$

From the estimates (1) and (2) for the Bessel kernel, we deduce

$$G_\alpha * \mu(x) \le I_{\alpha,1} * \mu(x) + C\int\limits_{R^N} e^{-\frac{|x-y|}{2}}d\mu(y).$$

We pose $I(x) = \int\limits_{R^N} e^{-\frac{|x-y|}{2}}d\mu(y)$. We estimate this integral.

If $E(x) = e^{-\frac{x}{2}}$, then $I(x) = E * \mu(x)$.

We consider the function $\phi$ defined by $\phi(x) = |B(0,1)|^{-1}$ if $|x| \le 1$, and $\phi(x) = 0$, if $|x| > 1$.

Then, there is a constant $C$ such that $E \le CE * \phi$.

This implies $E * \mu \le CE * \phi * \mu \le CE * M_{\alpha,1}\mu$, since $\phi * \mu \le M_{\alpha,1}\mu$.

By [10 or 21], we get $|||E * \mu|||_A \le C|||E * M_{\alpha,1}\mu|||_A \le C||E||_1|||M_{\alpha,1}\mu|||_A$.

This gives (6).

Hence, for obtaining the desired result, it suffices to prove that

$$|||I_{\alpha,1} * \mu|||_A \leq C|||M_{\alpha,1}\,\mu|||_A.$$

For this goal, recall the so called " *modified good $\lambda$ inequality*" (see [2, Chapter 3, Th. 3.6.2]):

*There are* $\tilde{a} > 1$ and $b > 0$ such that for any $\lambda > 0$ and any $\epsilon$, $0 < \epsilon \leq 1$,

$$|\{x : I_{\alpha,1} * \mu(x) > \tilde{a}\lambda\}| \leq b\epsilon^{\frac{N}{N-\alpha}}\,|\{x : I_{\alpha,1} * \mu(x) > \lambda\}| + |\{x : M_{\alpha,1}\mu(x) > \epsilon\lambda\}|.$$

We proceed by the same method as above and obtain 2.

This completes the proof. ■

## 4. QUASICONTINUITY

We recall the general definition of quasicontinuity.

**Definition 2** *Let $C$ be a capacity on $R^N$ and let $f$ be a function defined $C-$quasieverywhere on $R^N$ or on some open subset of $R^N$. Then $f$ is said to be $C-$quasicontinuous if for every $\epsilon > 0$, there is an open set $O$ such that $C(O) < \epsilon$ and $f\,|_{O^c} \in C(O^c)$.*

*In other words, the restriction of $f$ to the complement of $O$ is continuous in the induced topology.*

For Bessel capacity $B'_{m,A}$, we write $(m, A)$-quasicontinuous in place of $B'_{m,A}$-quasicontinuous.

**Theorem 2** *Let $A$ be an N-function satisfying the $\Delta_2$ condition. If $f \in \mathbf{L}_A$, then the potential $G_m * f$, $m > 0$, is $(m, A)-$quasicontinuous.*

*Hence every element in $\mathbf{L}_{m,A}$ has an $(m, A)-$quasicontinuous representative.*

*Proof.* By [6, Théorème 3] we know that $G_m * f(x)$ is well defined and finite $(m, A) - q.e.$ Since $A$ verifies the $\Delta_2$ condition, there is a sequence $(f_i)_i$ of functions in $\mathbf{D}$ which converges to $f$ in $\mathbf{L}_A$. ( Here $\mathbf{D} = \mathbf{D}(R^N)$ is the space of $C^\infty$ functions with compact support in $R^N$). From [6, Théorème 4] there is a subsequence $(f'_i)_i$ of the sequence $(f_i)$, such that $G_m * f'_i \to G_m * f$ $(m, A) - q.u.$

Hence $\forall\epsilon > 0$, $\exists X : B_{m,A}(X) < \frac{\epsilon}{2}$, and $G_m * f'_i \to G_m * f$ uniformly on ${}^cX$.

Since $B_{m,A}$ is outer (see [9, Théorème 2], there is an open set $O$, which contains $X$, such that

$$B_{m,A}(O) < \epsilon \quad \text{and } G_m * f'_i \to G_m * f \text{ uniformly on } {}^cO.$$

Since the elements of the sequence $(G_m * f_i)_i$ are continuous, $G_m * f$ is continuous on ${}^cO$. This achieves the proof. ■

We denote by $\chi$ the normalized characteristic function of the unit ball. For $r > 0$, we define $\chi_r$ by $\chi_r(x) = r^{-N}\chi(\frac{x}{2})$.

**Theorem 3** *Let $A$ be any N-function. Let $f = G_m * g \in \mathbf{L}_{m,A}$, $m > 0$. Then*

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy = G_m * g(x)$$

*wherever* $G_m * |g|(x) < \infty$, *i.e.*, $(m,A)-q.e.$

*Proof.* It is easily seen, by the estimates for Bessel kernel, that there is a constant $K$ such that $\chi_r * G_m \leq KG_m$ for all $r \leq 1$. In fact, if $|x| \leq 2$ and $r \leq \frac{|x|}{2}$, then by (1)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} G_m(y)dy \leq G_m\left(\tfrac{x}{2}\right) \leq KG_m(x).$$

On the other hand, if $|x| \geq 2$ and $r \leq 1$, then by (2)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} G_m(y)dy \leq \max_{|x-y|\leq 1} G_m(y) \leq KG_m(x).$$

And if $|x| \leq 2$ and $r \geq \frac{|x|}{2}$, then by (1)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} G_m(y)dy \leq \frac{1}{|B(x,r)|}\int_{B(x,3r)} G_m(y)dy \leq Kr^{m-N}$$
$$\leq K\left(\tfrac{|x|}{2}\right)^{m-N} \leq KG_m(x).$$

The Lebesgue dominated convergence theorem gives

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy = \lim_{r\to 0}\int_{B(x,r)} (\chi_r * G_m)(y)g(x-y)dy = G_m * g(x),$$

when $G_m * |g|(x) < \infty$. The proof is complete. ■

**Theorem 4** *Let $A$ be any N-function. Let $f_1$ and $f_2$ be two $(m,A)-$quasicontinuous functions, $m > 0$. Suppose that $f_1(x) = f_2(x)$ almost everywhere. Then*

$$f_1(x) = f_2(x)\ (m,A)\text{–quasieverywhere.}$$

*Proof.* Suppose that $f = f_1 - f_2$ is $(m,A)-$quasicontinuous and $f(x) = 0$ almost everywhere. We must show that $f(x) = 0$ $(m,A)-$quasieverywhere.

By definition of quasicontinuity there are open sets $(O_n)_n$ such that

$$\lim_{n\to\infty} B_{m,A}(O_n) = 0 \text{ and } f\,|^c\,(O_n) \in \mathbf{C}\left({}^c(O_n)\right).$$

Hence, there is a sequence $(\Psi_n)_n \subset \mathbf{L}_A^+$ such that

$$\lim_{n\to\infty} |||\Psi_n|||_A = 0 \text{ and } \varphi_n(x) = G_m * \Psi_n(x) \geq 1 \text{ on } O_n.$$

By [6, Théorème 4] we can suppose that $\lim_{n\to\infty} \varphi_n(x) = 0$ $(m,A)-q.e.$

Let $x$ be such a point. By the previous theorem, we can suppose, for all $n$, that

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\varphi_n(y)dy=\varphi_n(x).$$

Thus, for a large $n$, we have

$$\limsup_{r\to 0}\frac{|O_n\cap B(x,r)|}{|B(x,r)|}\leq\limsup_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)\cap O_n}\varphi_n(y)dy$$

$$\leq\limsup_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\varphi_n(y)dy=\varphi_n(x)<1.$$

This implies that $F_r = B(x,r)\setminus O_n$ has a strictly positive Lebesgue measure for all $r>0$. By hypothesis, this means that there is $y_r\in F_r$, for all $r>0$, such that $f(y_r)=0$. But $x\in^c(O_n)$ and $f\in \mathrm{C}\left(^c(O_n)\right)$, show that $f(x)=0$. The proof is complete. ■

Let $f\in \mathbf{L}_{m,A}$, $m>1$ and suppose that $A$ satisfies the $\Delta_2$ condition. Then $f$ is $(m,A)-$quasicontinuous. Denote by $Df$ any partial derivative of order 1 of $f$, taken in the distributional sense. Then $Df\in \mathbf{L}_{m-1,A}$, and hence $Df$ is $(m-1,A)-$quasicontinuous.

**Corollary 1** *Let $A$ be an N-function satisfying the $\Delta_2$ condition. Let $f,g\in \mathbf{L}_{m,A}$, $m>1$. Then*

$$D(fg)=f(x)Dg(x)+Df(x)g(x)\ (m-1,A)-q.e. \tag{7}$$

*Proof.* The derivatives of $f$, $g$ and $fg$, in the distributional sense, are almost everywhere equal to the pointwise derivatives. Hence the two sides of (7) are almost everywhere equal, by the ordinary formula of the derivative of a product. But the two sides are $(m-1,A)-$quasicontinuous, because the sum and the product of quasicontinuous functions are quasicontinuous.

This completes the proof. ■

We extend the last theorem for any kernel and with a less restrictive property that the quasicontinuity; but for reflexive Orlicz space. This is the tribute to pay!

By $\mathbf{B}(R^N)$ we note the family of Borelian sets in $R^N$.

**Theorem 5** *Let $k$ be any kernel and suppose that the N-function $A$ is such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Let $g_1$ and $g_2$ be two functions verifying the following: $\forall\epsilon>0$, $\exists X\in\mathbf{B}(R^N)$:$C_{k,A}(X)<\epsilon$ and the restrictions of $g_1$ and $g_2$ to $^cX$ are continuous.*

*Suppose that $\{x:g_1(x)\neq g_2(x)\}\in\mathbf{B}(R^N)$ and that $g_1(x)=g_2(x)$ a.e. Then*

$$g_1(x)=g_2(x)\ (k,A)-q.e.$$

*Proof.* We follow Sjödin's method in [22] relative to the $L^p$ Lebesgue spaces case.

We pose $E=\{x:g_1(x)\neq g_2(x)\}$ and suppose $C_{k,A}(E)=c>0$. We suppose also $c$ finite. Let $\epsilon>0$ be such that $c>2\epsilon$ and $E_1\in\mathbf{B}(R^N)$ be such that $C_{k,A}(E_1)<\epsilon$ with the restrictions of $g_1$ and $g_2$ to $^cE_1$ continuous. By [7, Théorème 1] $C'_{k,A}$ is subadditive. Hence $C'_{k,A}\left(E\setminus E_1\right)>c-\epsilon$.

By [5, Théorème 3] $E \setminus E_1$ is capacitable. Thus, there is a compact $K \subset E \setminus E_1$, such that $C'_{k,A}(K) > c - \epsilon$.

By [5, Théorème 4] we have the equality $C'_{k,A}(K) = D_{k,A}(K)$, and $K$ has a distributional measure $\mu$, for $D_{k,A}$. Hence $\|\mu\| = C'_{k,A}(K)$.

Let $\phi \in \mathbf{D}(R^N)$ be with support in the unit ball and such that $\|\phi\|_1 = 1$. We pose $\phi_i(x) = i^N \phi(ix)$, $\mu_i = \mu * \phi_i$ $(i = 1, 2, ...)$, and $K_i = \left\{x \in R^N : dist(x, K) \leq i^{-1}\right\}$.

This implies that $\mu_i$ is a test measure for $D_{k,A}(K_i \setminus E)$ since it is absolutely continuous with respect to the Lebesgue measure $\mathbf{m}$, and $\mathbf{m}(E) = 0$, and on the other hand we have by [21] (see also [10] for a simple proof)

$$\|k * \mu_i\|_{A^*} = \|k * \mu * \phi_i\|_{A^*} \leq \|k * \mu\|_{A^*} \|\phi_i\|_1 .$$

We get $C'_{k,A}(K_i \setminus E) = D_{k,A}(K_i \setminus E) \geq \|\mu_i\| = \|\mu\| = C'_{k,A}(K) > c - \epsilon$.

Hence $C'_{k,A}\left[(K_i \setminus E) \setminus E_1\right] > c - 2\epsilon > 0$, which implies $(K_i \setminus E) \setminus E_1 \neq \emptyset$, for $i = 1, 2, ...$

Let $(x_p)_p$ be a sequence in $(K_i \setminus E) \setminus E_1$ and $(y_p)_p$ be another sequence in $K$ such that $|x_p - y_p| \leq p^{-1}$, $p = 1, 2, \dots$ .

Since $K$ is a compact set, there is $y \in K$ such that $\lim\limits_{p \to \infty} y_p = y$. This implies $\lim\limits_{p \to \infty} x_p = y$. We have

$|g_1(y) - g_2(y)| \leq |g_1(y) - g_1(x_p)| + |g_1(x_p) - g_2(x_p)| + |g_2(x_p) - g_2(y)|$, $p = 1, 2, \dots$

But $g_1(x_p) - g_2(x_p) = 0$, and by continuity we get

$$|g_1(y) - g_1(x_p)| \to 0, \text{ and } |g_2(x_p) - g_2(y)| \to 0 \text{ as } p \to \infty.$$

Thus $g_1(y) = g_2(y)$, and we have a contradiction.

The case $c = \infty$ reduces easily to the precedent one. The proof is finished. ■

## 5. OPERATIONS ON POTENTIALS

We need the following lemma.

**Lemma 1** *Let $A$ be any $N$-function. For all multiindices $\xi$ such that $|\xi| < m < N$, there is a constant $C$ such that for all $f \in \mathbf{L}_A$ and for almost every $x$,*

$$\left|D^{\xi}(G_m * f)(x)\right| \leq C Mf(x)^{\frac{|\xi|}{m}} \left(G_m * |f|(x)\right)^{1 - \frac{|\xi|}{m}} .$$

*Proof.* The proof is exactly the same as in [2, Chapter 3, Proposition 3.1.7] for the $L^p$ Lebesgue spaces case.

If $Mf(x) = \infty$, there is nothing to prove. Consider the case $Mf(x) < \infty$. By (1), (2) and the estimate

$$\int\limits_{|x-y|<d} \frac{|f(y)|\, dy}{|x-y|^{N-m}} \leq D d^m Mf(x), \tag{8}$$

where $D$ is a constant, there is a constant $C'$ such that

$$G_m * |f|(x) \leq C' \int\limits_{|x-y|<1} \frac{|f(y)|dy}{|x-y|^{N-m}} + C' \int\limits_{|x-y|\geq 1} |f(y)|\, e^{-\frac{|x-y|}{2}}\, dy$$

$$\leq C'Mf(x) + C' \sum_{i=1}^{\infty} e^{-\frac{i}{2}} \int\limits_{i\leq|x-y|<i+1} |f(y)|\, dy$$

$$\leq C'Mf(x) + C'Mf(x) \sum_{i=1}^{\infty} (i+1)^N e^{-\frac{i}{2}} = CMf(x).$$

Hence by (4), (5), (2) and (8), we get for every $\delta \leq 1$

$$\left|D^{\xi}(G_m * f)(x)\right| \leq C \int\limits_{|x-y|<d} \frac{|f(y)|dy}{|x-y|^{N-m+|\xi|}} + \int\limits_{d\leq|x-y|<1} \frac{|f(y)|dy}{|x-y|^{N-m+|\xi|}} +$$

$$+C' \int\limits_{|x-y|\geq 1} G_m(x-y)\, |f(y)|\, dy.$$

Whence $\left|D^{\xi}(G_m * f)(x)\right| \leq C\left(\delta^{m-|\xi|} Mf(x) + \delta^{-|\xi|} G_m * |f|(x) + G_m * |f|(x)\right)$.

We choose $\delta^m = \dfrac{G_m * |f|(x)}{CMf(x)}$. Hence $\delta \leq 1$, and we obtain the lemma. ■

The $L^p$ version of the following Theorem can be found in [1] for $m \in R$ such that $0 < m < N$.

**Theorem 6** *Let $m$ be an integer such that $0 < m < N$ and $A$ be an $N$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Let $k$ be an integer such that $k \geq m$ and $T \in \mathbf{C}^k(R^+)$ verifies the following condition*

$$\sup\left|x^{i-1} T^{(i)}(x)\right| \leq L < \infty,\ i = 1, 2, \ldots, k.$$

*Then $T \circ (G_m * f) \in \mathbf{L}_{m,A}$, for all $f \in \mathbf{L}_A^+$, and there is a constant $C$, which depends only on $A$, $m$ and $N$, such that*

$$|||T \circ (G_m * f)|||_{m,A} \leq CL|||G_m * f|||_{m,A} = CL|||f|||_A.$$

*Proof.* We begin by the case where $f \in \mathbf{D}(R^N)$ and $f \geq 0$. Let $u = G_m * f$; then $u$ is positive and hence $T \circ u$ is well defined. Let $\xi$ be any multiindice such that $|\xi| = m$. By the calculus of partial derivatives of the compositions we obtain

$$D^{\xi}(T \circ u) = \sum_{i=1}^{m} T^{(i)} \circ u \sum c_{\xi} D^{\xi_1} u \ldots D^{\xi_i} u,$$

where the sum is for the multiindices $\{\xi_1, \ldots, \xi_i\}$ such that $\xi_1 + \ldots + \xi_i = \xi$ and $\left|\xi_j\right| \geq 1$ $\forall j$. The value of the coefficients $c_{\xi}$ is of no importance.

By hypothesis $\left|D^{\xi}(T \circ u)\right| \leq CL \sum_{i=1}^{m} u^{1-i} \sum \left|D^{\xi_1} u \ldots D^{\xi_i} u\right|$.

For $i > 1$, we estimate these derivatives with the aid of precedent Lemma. We get $\left|D^{\xi_j}u\right| CMf^{\frac{|\xi_j|}{m}}u^{1-\frac{|\xi_j|}{m}}$.

Since $\sum_{j=1}^{i}\left(1-\frac{|\xi_j|}{m}\right) = i - \frac{|\xi|}{m} = i-1$, we obtain

$$\sum_{i=2}^{m} u^{1-i}\sum\left|D^{\xi_1}u \ldots D^{\xi_i}u\right| \leq C\sum_{i=2}^{m} u^{1-i}Mf\, u^{i-1} = CMf.$$

By adding the term for $i=1$, we get $\left|D^{\xi}(T\circ u)\right| \leq CL\left(|Mf| + \left|D^{\xi}u\right|\right)$.

From [10] we know that $|||Mf|||_A \leq C|||f|||_A$, and that

$$|||D^{\xi}(G_m * f)|||_A \leq C|||f|||_A, \text{ for } |\xi| = m.$$

This achieve the proof for $f \in \mathbf{D}(R^N)$.

Let us treat the general case when $f \in \mathbf{L}_A^+$. Since $A$ verifies the $\Delta_2$ condition, there is a sequence of positive functions $f_i \in \mathbf{D}(R^N)$, $i = 1, 2, \ldots$, such that $\lim_{i\to\infty} |||f_i - f|||_A = 0$.

Hence $|||T\circ(G_m * f_i)|||_{m,A} \leq CL|||f|||_A$ for sufficiently large $i$.

That if we pose $T\circ(G_m * f_i) = G_m * g_i$, we can assume that the sequence $(g_i)_i$ converges weakly in $\mathbf{L}_A$ to an element $g$, with $|||g|||_A \leq CL|||f|||_A$.

We have to show that $G_m * g = T\circ(G_m * f)$.

From [10 or 21], $||G_m * (f - f_i)|||_A \leq || G_m ||_1\, |||f - f_i|||_A$.

This implies that the sequence $(G_m * g_i)_i$ converges strongly in $\mathbf{L}_A$ to $G_m * f$. By considering a subsequence, we can assume that $\lim_{i\to\infty} G_m * f_i = G_m * f$ a.e.

The continuity of $T$ gives $\lim_{i\to\infty} G_m * g_i = T\circ(G_m * f)$ a.e.

The weak convergence of the sequence $(g_i)_i$ implies that $G_m * g$ is the limit of the sequence $(G_m * g_i)_i$. In fact, pose for any $i$, $h_i = g_i - g$. Then for any $\epsilon > 0$, we have

$$G_m * h_i(x) = \int\limits_{|x-y|<\epsilon} G_m(x-y)h_i(y)dy + \int\limits_{|x-y|>\epsilon} G_m(x-y)h_i(y)dy.$$

Since $G_m \in \mathbf{L}_{A^*}$ out the origin, we have by Hölder inequality in Orlicz spaces

$$\lim_{i\to\infty} \int\limits_{|x-y|>\epsilon} G_m(x-y)h_i(y)dy = 0.$$

By Fatou's Lemma we get

$$\begin{aligned} |||\lim_{i\to\infty} G_m * h_i|||_A &\leq \liminf_{i\to\infty} |||\int\limits_{|y|<\epsilon} G_m(y)\mid h_i(.-y)\mid dy|||_A \\ &\leq \sup_i |||h_i|||_A \int\limits_{|y|<\epsilon} G_m(y)dy, \end{aligned}$$

which is arbitrarily small.

Hence for almost all $x$,

$$G_m * g(x) = \lim_{i\to\infty} G_m * g_i(x) = \lim_{i\to\infty} T\circ(G_m * f_i)(x) = T\circ(G_m * f)(x).$$

The proof is finished. ■

## 6. OTHER DEFINITION OF CAPACITY AND REMOVABLE SINGULARITIES

We begin by the following definitions.

**Definition 3** *For $X \subset R^N$, we pose*

$$N_{k,A}(X) = \inf\{A(|||\varphi|||_{k,A}) : \varphi \in \mathbf{S} \text{ and } \varphi = 1 \text{ in a neighborhood of } X\}$$
$$N'_{k,A}(X) = \inf\{|||\varphi|||_{k,A} : \varphi \in \mathbf{S} \text{ and } \varphi = 1 \text{ in a neighborhood of } X\}.$$

*Here $\mathbf{S} = \mathbf{S}(R^N)$ is the Schwartz space of rapidly decreasing functions.*

If $k = G_m$, we write $N_{m,A} = N_{G_m,A}$.

**Definition 4** *Let $K \subset R^N$ be a compact set, and let $\mathbf{P}$ a partial differential operator defined in a neighborhood of $K$. Then $K$ is said to be removable for $\mathbf{P}$ in $\mathbf{L}_A$ if any solution $v$ of $\mathbf{P}v = 0$ in $O\setminus K$ for some bounded open neighborhood of $K$, such that $v \in \mathbf{L}_A(O\setminus K)$, can be extended to a function $\tilde{v} \in \mathbf{L}_A(O)$ such that $\mathbf{P}\tilde{v} = 0$ in $O$.*

We give a result that can be proved, as in the Lebesgue spaces case, without going deeply into the theory of partial differential equations.

**Theorem 7** *Let $m$ be an integer such that $0 < m < N$. Let $K \subset R^N$ be a compact set, and let $\mathbf{P}$ an elliptic linear partial differential operator of order $m$ with constant coefficients. Let $A$ be an N-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Then $K$ is not removable for $\mathbf{P}$ in $\mathbf{L}_A$ if $B_{m,A^*}(K) > 0$, and it is removable if $N_{m,A^*}(K) = 0$.*

*Proof.* Assume that $B_{k,A^*}(K) > 0$. Then by [5, Théorème 4] there is a nonzero $\mu \in M^+(K)$ such that $G_m * \mu \in \mathbf{L}_A$. Let $S$ be the fundamental solution of $\mathbf{P}$. By the properties of fundamental solutions of elliptic linear operators, there is a constant $C$ such that $|S(x)| \le C|x|^{m-N}$ for small $|x|$. See [14, pp 61-65]. It follows that $S * \mu \in \mathbf{L}_{A,loc}$. Moreover, $S * \mu$ is a solution of $\mathbf{P}v = 0$ in ${}^cK$, which proves that $K$ is not removable.

On the other hand, assume that $N_{m,A^*}(K) = 0$. Then $|K| = 0$, otherwise we could use Lebesgue measure restricted to $K$ in the first part of the proof. Hence, a given solution $v$ in $O\setminus K$ is defined *a.e.* in $O$, so it can be considered as a distribution in $O$.

Let $\epsilon > 0$ and let $\chi \in \mathbf{S}$ be such that $\chi = 1$ in a neighborhood of $K$, and satisfy $|||\chi|||_{m,A^*} < \epsilon$. Let $\varphi \in \mathbf{D}(O)$. We claim that

$$\int v\mathbf{P}^*\varphi dx = 0, \tag{9}$$

$\mathbf{P}^*$ denoting the adjoint operator of $\mathbf{P}$.

We have $(1-\chi)\varphi \in \mathbf{D}(O \setminus K)$, and hence $\int v\mathbf{P}^*\left((1-\chi)\varphi\right) dx = 0$.

Hölder inequality in Orlicz spaces gives

$$\left|\int v\mathbf{P}^*\varphi dx\right| = \left|\int v\mathbf{P}^*(\chi\varphi) dx\right| \leq ||v||_{A,O}|||\mathbf{P}^*(\chi\varphi)|||_{A^*}.$$

The Leibniz formula and the equivalence of norms in $\mathbf{L}_{m,A^*}$ and $W^m\mathbf{L}_{A^*}$ give a constant $C$ such that $|||\mathbf{P}^*(\chi\varphi)|||_{A^*} \leq C\epsilon$. Since $\epsilon$ is arbitrary, (9) follows. Thus $v$ is a weak solution in $O$, and we obtain the theorem by the regularity theory for elliptic equations, see Hörmander [12, Theorem 4.4.1]. ■

We remark the immediate inequality $B_{m,A}(X) \leq N_{m,A}(X)$. In view of the last theorem, it is of considerable interest that these set functions are in fact equivalent.

**Theorem 8** *Let $m$ be an integer such that $0 < m < N$, and $A$ be an $N$-function such that $A$ and $A^*$ satisfy the $\Delta_2$ condition. Then there is a constant $C$ such that for all $X \subset R^N$*

$$B_{m,A}(X) \leq N_{m,A}(X) \leq CB_{m,A}(X).$$

*This means that a compact $K \subset R^N$, is removable in $\mathbf{L}_A$ for an elliptic linear operator of order $m$ with constant coefficients if and only if $B_{m,A}(K) = 0$.*

*Proof.* Let $\epsilon > 0$. Then since $B_{m,A}$ is outer, there is $f \in \mathbf{L}_A^+$ such that $G_m * f \geq 1$ on a neighborhood of $X$, and $A(|||f|||_A) \leq B_{m,A}(X) + \epsilon$.

Let $T \in \mathbf{C}^\infty(R)$ be a function such that $0 \leq T \leq 1$, $T(x) = 0$ for $0 \leq x \leq \frac{1}{2}$ and $T(x) = 1$ for $1 \leq x$. An application of Theorem 1, §5 gives the desired result. ■

The first $L^p$ version of this theorem was been proved by V.G. Maz'ya [18, Chapter 9.3]. The $L^p$ version for general $m$ is due to D.R. Adams and J.C. Poking [3]. The general case when $m \in R$ is such that $0 < m < N$, remains open.

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