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AN ESTIMATE FOR THE RATE OF APPROXIMATION OF FUNCTIONS BY CHEBYSHEV POLYNOMIALS

KAMEL AL-KHALED

Abstract. *For a function $f(x)$ to be of bounded variation on [-1,1] and $C_n(f;x)$ the n^{th} partial sum of the expansion of $f(x)$ in a Chebyshev series of the second kind, we find an estimate for the rate of convergence of the sequence $\{C_n(f;x)\}$ to $\frac{1}{2}[f(x+0)+f(x-0)]$.*

1. STATEMENT OF THE RESULT

Let be $f(x)$ be a real-valued function defined on [-1,1]. Let $U_n(x)$ be the Chebyshev polynomial of the second kind that is defined by

$$U_n(x) = \frac{\sin(n+1)arccosx}{\sin(arccosx)}$$

For $f(x)$ to be a function of bounded variation on [-1,1] and $C_n(f;x)$ the n^{th} partial sum of the expantion of $f(x)$ in a Chebyshev series of the second kind defined as

$$C_n(f;x) = \sum_{n=0}^{\infty} A_n U_n(x) \tag{1.1}$$

where

$$A_n = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-t^2} f(t) U_n(t) dt, n = 0, 1, 2, \ldots \tag{1.2}$$

The sequence $\{C_n(f;x)\}$ converges to $f(x)$ pointwise on (-1,1) and uniformly on every closed subinterval of (-1,1) wherever $f(x)$ is a continuous function on [-1,1]. At the end points -1 and 1, however the sequence diverges.

Bojanic, B. and Cheng, F. H. [3] have studied the behavior of Hermite-Fejer polynomials $H_n(f;x)$ for functions of bounded variation on [-1,1] by taking the interpolation over the zeros of Chebyshev polynomials of first kind. They gave an estimate for the rate of convergence of $H_n(f;x)$ at points of continuity of f and proved that at points of discontinuity where $f(x+0) \neq f(x-0)$ the $H_n(f;x)$ diverge. Al-Jarrah [2] studied the same problem where the interpolation is taken over the zeros of the Chebyshev polynomials of the second kind. We give here a similar treatment of convergence problem where $H_n(f;x)$ (see[2]) is replaced by $C_n(f;x)$ as defined by (1.1) and (1.2). We now formulate our main theorem in this paper.

Theorem1.1 Let $f(x)$ be a function of bounded variation on [-1,1] and continuos at $x \in (-1,1)$.

Let

$$g_x(t) = \begin{cases} f(t) - f(x+0), & \text{if } x < t \le 1 \\ 0, & \text{if } x = t \\ f(t) - f(x-0), & \text{if } -1 \le t < x \end{cases}$$

Then for every $x \in (-1,1)$ and $n \ge 2$ we have

$$\left|C_n(f;x) - \frac{1}{2}(f(x+0) + f(x-0))\right| \le \frac{9}{n\sqrt{1-x^2}}\Big(\frac{1}{1+x}\sum_{k=1}^{n} V_{g_x}[x - \frac{1+x}{k}, x] + $$
$$+ \frac{1}{1-x}\sum_{k=1}^{n} V_{g_x}[x, x + \frac{1-x}{k}]\Big) + \frac{4}{n\pi\sqrt{1-x^2}}\,|f(x+0) - f(x-0)|$$

Here, $V_{g_x}[a,b]$ is the total variation of $g_x(t)$ on [a,b]. It is easy to see that g_x is continuous at $t = x$ and of bounded variation on [-1,1].

2. PROOF OF THE THEOREM

We know from (1.1), (1.2) that

$$\begin{aligned} C_n(f;x) &= \sum_{n=0}^{\infty} \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-t^2} f(t) U_n(t) U_n(x) dt \\ &= \int_{-1}^{1} \sqrt{1-t^2} f(t) V_n(x,t) dt \end{aligned} \tag{2.1}$$

where

$$V_n(x,t) = \frac{2}{\pi}\sum_{k=0}^{n} U_k(x) U_k(t) \tag{2.2}$$

Now we call

$$k_1 = \int_{-1}^{x - \frac{1+x}{n}} \sqrt{1-t^2} g_x(t) V_n(x,t) dt$$

$$k_2 = \int_{x - \frac{1+x}{n}}^{x + \frac{1-x}{n}} \sqrt{1-t^2} g_x(t) V_n(x,t) dt$$

$$k_3 = \int_{x + \frac{1-x}{n}}^{1} \sqrt{1-t^2} g_x(t) V_n(x,t) dt$$

$$k_4 = f(x-0)\Big[\int_{-1}^{x} \sqrt{1-t^2} V_n(x,t) dt - \frac{1}{2}\Big]$$

and

$$k_5 = f(x+0)\Big[\int_{x}^{1} \sqrt{1-t^2} V_n(x,t) dt - \frac{1}{2}\Big]$$

and so by the definition of $g_x(t)$ and (2.1), for any fixed $x \in (-1,1)$ we have

$$C_n(f;x) - \frac{1}{2}(f(x+0) + f(x-0)) = k_1 + k_2 + k_3 + k_4 + k_5$$

To find an estimate for each of the above quantities k_1, k_2, k_3, k_4 and k_5, we proceed as follows.

By (2.2), the integral in k_4 can be written as

$$\int_{-1}^{x}\sqrt{1-t^2}V_n(x,t)dt=(\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}dt)+(\frac{2}{\pi}\sum_{k=1}^{n}U_k(x)\int_{-1}^{x}\sqrt{1-t^2}U_k(t)dt) \quad (2.3)$$

evaluating the first integral in the left hand side of (2.3) to get

$$\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}dt=\frac{1}{\pi}x\sqrt{1-x^2}+\frac{1}{\pi}\arcsin x+\frac{1}{2}$$

or, for $x=\cos\theta$, we have

$$\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}dt=1-\frac{\theta}{\pi}+\frac{\sin 2\theta}{2\pi} \quad (2.4)$$

also the second integral in the right hand side of (2.3) has the value

$$\int_{-1}^{x}\sqrt{1-t^2}U_k(t)dt=\frac{1}{2}(\frac{\sin(k+2)\theta}{k+2}-\frac{\sin k\theta}{k})$$

and so the second term in the left hand side of (2.3) becomes

$$\begin{aligned}\frac{2}{\pi}\sum_{k=1}^{n}U_k(x)\int_{-1}^{x}\sqrt{1-t^2}U_k(t)dt &= \frac{-1}{\pi}\sum_{k=3}^{n}\frac{\sin 2k\theta}{k}+\frac{\sin(n+1)\theta\sin n\theta}{(n+1)\pi\sin\theta}+\\ &+\frac{\sin(n+2)\theta\sin(n+1)\theta}{(n+2)\pi\sin\theta}-\frac{\sin 2\theta}{\pi}-\frac{\sin 3\theta\cos\theta}{\pi} \quad (2.5)\end{aligned}$$

Now substitute the well known fact

$$\arccos x=\frac{\pi}{2}-\sum_{k=1}^{\infty}\frac{\sin 2k(\arccos x)}{k}$$

or, for $\theta=\arccos x$, $0<\theta<\pi$

$\theta=\frac{\pi}{2}-\sum_{k=1}^{\infty}\frac{\sin 2k\theta}{k}$, substitute this into Equation (2.4) to obtain

$$\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}dt=\frac{1}{2}+\frac{3\sin 2\theta}{2\pi}+\frac{\sin 4\theta}{2\pi}+\frac{1}{\pi}\sum_{k=3}^{\infty}\frac{\sin 2k\theta}{k} \quad (2.6)$$

Using (2.5), (2.6). Equation (2.3) takes the form

$$\int_{-1}^{x}\sqrt{1-t^2}V_n(x,t)dt=\frac{1}{2}+\frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{\sin 2k\theta}{k}+\frac{\sin(n+1)\theta\sin n\theta}{\pi(n+1)\sin\theta}+\frac{\sin(n+2)\theta\sin(n+1)\theta}{\pi(n+1)\sin\theta}$$

Therefore for $0<\theta<\pi$

$$k_4=f(x-0)\Big[\frac{1}{\pi}\sum_{k=n+1}^{\infty}\frac{\sin 2k\theta}{k}+\frac{\sin(n+1)\theta\sin n\theta}{\pi(n+1)\sin\theta}+\frac{\sin(n+2)\theta\sin(n+1)\theta}{\pi(n+1)\sin\theta}\Big]$$

and

$$k_5=f(x+0)\Big[\frac{-1}{\pi}\sum_{k=n+1}^{\infty}\frac{\sin 2k\theta}{k}-\frac{\sin(n+1)\theta\sin n\theta}{\pi(n+1)\sin\theta}-\frac{\sin(n+2)\theta\sin(n+1)\theta}{\pi(n+1)\sin\theta}\Big].$$

To find abound for $|k_4 + k_5|$, notice that

$$\left|\sum_{k=n+1}^{\infty} \frac{\sin 2k\theta}{k}\right| \leq \frac{2}{n \sin\ \theta}, 0 < \theta < \pi$$

i.e., for $0 < \theta < \pi, -1 < x < 1$, and $n \geq 1$, we have

$$|k_4 + k_5| \leq \frac{4}{n\pi\sqrt{1-x^2}}\, |f(x+0) - f(x-0)| \tag{2.7}$$

Next, to find an estimate for k_2, note that

$$|V_n(x,t)| \leq \frac{2(n+1)}{\pi\sqrt{(1-x^2)(1-t^2)}}$$

For $t \in [x - \frac{1+x}{n}, x + \frac{1-x}{n}]$ and since $g_x(x) = 0$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leq V_{g_x}[x - \frac{1+x}{n}, x + \frac{1-x}{n}]$$

and so, for $-1 < x < 1, n \geq 2$ we have

$$|k_2| \leq \frac{6}{\pi\sqrt{1-x^2}} V_{g_x}[x - \frac{1+x}{n}, x + \frac{1-x}{n}] \tag{2.8}$$

Finally we estimate k_1 and k_3.

By (2.2) and a known formula, [6, p. 70] we have

$$V_n(x,t) = \frac{\sin(n+2)\arccos x.\sin(n+1)\arccos t - \sin(n+2)\arccos t.\sin(n+1)\arccos x}{\pi(x-t)\sin\arccos x.\sin\arccos t}$$

set

$$k_n(x,y) = -\int_y^1 \sqrt{1-t^2}V_n(x,t)dt, (x < y \leq 1) \tag{2.9}$$

Applying the mean-value theorem, we can find a $\xi \in [y, 1]$ such that wih $x = \cos\theta$ we have

$$\begin{aligned} k_n(x,y) &= \frac{1}{\pi(y-x)}\Big[\frac{\sin(n+2)\theta}{\sin\theta}\int_y^{\xi}\sqrt{1-t^2}\frac{\sin\ (n+1)\arccos t}{\sin\arccos t}dt - \\ &\quad \frac{\sin(n+1)\theta}{\sin\ \theta}\int_y^{\xi}\sqrt{1-t^2}\frac{\sin(n+2)\arccos t}{\sin\ \arccos t}dt\Big] \end{aligned}$$

It is easy to show that

$$\left|\int_y^{\xi}\sqrt{1-t^2}\frac{\sin(n+1)\arccos t}{\sin\arccos t}dt\right| \leq \frac{2}{n}$$

and similarly

$$\left|\int_y^{\xi}\sqrt{1-t^2}\frac{sin(n+2)\arccos t}{\sin\arccos t}dt\right| \leq \frac{2}{n+1}$$

so that $k_n(x,y)$ can be bounded by

$$|k_n(x,y)| \leq \frac{4}{n\pi(y-x)\sqrt{1-x^2}}$$

Also from (2.9) and $g_x(x) = 0$, we have

$$\begin{aligned}|k_3| &= \left|\int_{x+\frac{1-x}{n}}^{1} \sqrt{1-t^2} g_x(t) V_n(x,t)dt\right| = \left|\int_{x+\frac{1-x}{n}}^{1} g_x(t)\frac{\partial}{\partial t}k_n(x,t)dt\right| \\ &\leq \left|g_x(x+\frac{1-x}{k})k_n(x,x+\frac{1-x}{n})\right| + \int_{x+\frac{1-x}{n}}^{1} |k_n(x,t)|\, V'_{g_x}[x,t] \\ &\leq \frac{4}{\pi(1-x)\sqrt{1-x^2}} V_{g_x}[x,x+\frac{1-x}{n}] + \frac{4}{n\pi\sqrt{1-x^2}} \int_{x+\frac{1-x}{n}}^{1} \frac{1}{t-x} V'_{g_x}[x,t]\end{aligned}$$

But by a result to Bojanic [4, p. 77]

$$\int_{x+\frac{1-x}{n}}^{1} \frac{1}{t-x} V'_{g_x}[x,t] \leq \frac{1}{1-x}\left[V_{g_x}[x,1] + nV_{g_x}[x,x+\frac{1-x}{n}] + \sum_{k=1}^{n} V_{g_x}[x,x+\frac{1-x}{k}]\right]$$

we find that

$$|k_3| \leq \frac{16}{n\pi(1-x)\sqrt{1-x^2}} \sum_{k=1}^{n} V_{g_x}[x,x+\frac{1-x}{k}] \tag{2.10}$$

The same way we can find that

$$|k_1| \leq \frac{16}{n\pi(1+x)\sqrt{1-x^2}} \sum_{k=1}^{n} V_{g_x}[x-\frac{1+x}{k},x] \tag{2.11}$$

The proof of the theorem follows now from substituting the inequalities (2.7), (2.8), (2,10) and (2.11) into

$$\left|C_n(f;x) - \frac{1}{2}(f(x+0) + f(x-0))\right| \leq |k_1| + |k_2| + |k_3| + |k_4| + |k_5|.$$

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