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ON AN INTEGRAL INEQUALITY FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract. *Let g be an analytic function on the unit disc $U = \{z;\ |z| < 1\}$, with $g(0) = g'(0) - 1 = 0$, and let $f(z) = \int_0^z \frac{g(t)}{t} dt$. In [4], Petru T. Mocanu showed that if g satisfies the inequality $|g'(z) - 1| < \frac{8}{2+\sqrt{15}} = 1.362\ldots$, then $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$, which is equivalent to $Re \int_0^1 \frac{g(uz)}{ug(z)} du > \frac{1}{2}$, for $z \in U$. In this paper we extend the results under the additional condition $g''(0) = \ldots = g^{(n)}(0) = 0$.*

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1. INTRODUCTION AND PRELIMINARIES

Let A_n denote the class of functions of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$$

which are analytic in the unit disc $U = \{z;\ |z| < 1\}$.

In [3, Corollary 4.2] Petru T. Mocanu obtained the following result:

If $g \in A_1$ satisfies $|g'(z) - 1| < 1$, for $z \in U$, then

$$Re \int_0^1 \frac{g(uz)}{ug(z)} du > \frac{1}{2}, \quad \text{for } z \in U.$$

If we let

$$f(z) = \int_0^1 \frac{g(uz)}{u} du$$

then this last inequality is equivalent to

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \text{for } z \in U.$$

In [4], Petru T. Mocanu improved the above result, by showing that the same conclusion holds under the less restrictive condition $|g'(z) - 1| < \frac{8}{2+\sqrt{15}} = 1.362\ldots$.

In this paper we shall determine a positive number M_n so that if $g \in A_n$ and $|g'(z) - 1| < M_n$ then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \text{for } z \in U,$$

where f is given by

$$f(z) = \int_0^1 \frac{g(uz)}{u} du.$$

2. PRELIMINARIES

If f and g are analytic functions on U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

We shall use the following lemmas to prove our results.

Lemma 1. [1, p.192] *Let h be a convex function on U (i.e. h is univalent and $h(U)$ is a convex domain). If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is analytic in U and satisfies the differential subordination*

$$p(z) + zp'(z) \prec h(z),$$

then

$$p(z) \prec \frac{1}{nz^{1/n}} \int_0^z h(t) t^{1/n-1} dt.$$

Lemma 2. [2] *Let E be a set in the complex plane $\mathbf{C}$ and let q be an analytic and univalent function on U. Suppose that the function $H : \mathbf{C}^2 \times U \to \mathbf{C}$ satisfies*

$$H[q(\zeta), m\zeta q'(\zeta); z] \notin E$$

whenever $m \geq n$, $|\zeta| = 1$ and $z \in U$. If $p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is analytic on U, and satisfies

$$H[p(z), zp'(z); z] \in E, \quad \text{for } z \in U,$$

then $p \prec q$.

3. MAIN RESULTS

Theorem 1. *If $f \in A_n$ satisfies*

$$|f'(z) + zf''(z) - 1| < M, \quad z \in U, \tag{1}$$

where $M \leq M_n$, with

$$M_n = \frac{(n+1)^4}{n(n+3) + \sqrt{(n+1)^6 - 4n}}, \tag{2}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{3}$$

Proof. Since the inequality (1) can be rewritten as

$$f'(z) + zf''(z) \prec 1 + Mz, \quad z \in U$$

by using Lemma 1, we deduce $f'(z) \prec 1 + \frac{M}{n+1} z$ and

$$\frac{f(z)}{z} \prec 1 + \frac{M}{(n+1)^2} z. \tag{4}$$

Let $p(z) = \frac{zf'(z)}{f(z)}$ and $P(z) = \frac{f(z)}{z}$. Since (4) implies $P(z) \neq 0$, the function p is analytic in U and the inequality (1) becomes

$$|P(z)[zp'(z) + p^2(z)] - 1| < M, \quad z \in U. \tag{5}$$

The inequality (3) is equivalent to

$$p(z) \prec 1 + z \tag{6}$$

and in order to show that (6) holds, by Lemma 2, it is sufficient to check the inequality

$$|P(z)[m\zeta + (1+\zeta)^2 - 1| \geq M, \tag{7}$$

for all $m \geq n$, $|\zeta| = 1$ and $z \in U$.

If we let $\zeta = e^{i\theta}$

$$L(m, \theta, z) = |P(z)[m\zeta + (1+\zeta)^2] - 1|^2 = |P(z)\zeta(\zeta + \overline{\zeta} + m + 2) - 1|^2 =$$
$$= (2\cos\theta + m + 2)\{(2\cos\theta + m + 2)|P(z)|^2 - 2Re[e^{i\theta}P(z)]\} + 1.$$

From (4) we deduce

$$|P(z) - 1| < \frac{M}{(n+1)^2}$$

and

$$|P(z)| > 1 - \frac{M}{(n+1)^2}.$$

For $m \geq n$ we have

$$\frac{1}{2}\frac{\partial L}{\partial m} = (2\cos\theta + m + 2)|P(z)|^2 - Re[e^{i\theta}P(z)] =$$
$$= (m+2)|P(z)|^2 + Re\{e^{i\theta}P(z)[2\overline{P(z)} - 1] \geq$$
$$\geq |P(z)|[(n+2)|P(z)| - 2|P(z) - 1| - 1] \geq$$
$$\geq |P(z)|\left\{(n+2)\left[1 - \frac{M}{(n+1)^2}\right] - 2\frac{M}{(n+1)^2} - 1\right] \geq$$
$$\geq |P(z)|\left[(n+1) - \frac{(n+4)M}{(n+1)^2}\right] > 0,$$

which shows that L is an increasing function of m. Hence we deduce

$$L(m, \theta, z) \geq L(n, \theta, z) = (2\cos\theta + n + 2)\{(2\cos\theta + n + 2)|P(z)|^2 - 2Re[e^{i\theta}P(z)]\} + 1 =$$
$$= (2\cos\theta + n + 2)\{(n+2)|P(z)|^2 - 2Re[e^{i\theta}P(z)(\overline{P(z)} - 1)] + 1 \geq$$
$$\geq (2\cos\theta + n + 2)|P(z)|\{(n+2)|P(z)| - 2|P(z) - 1|\} + 1 \geq$$
$$\geq n\left[1 - \frac{M}{(n+1)^2}\right]\left\{(n+2)\left[1 - \frac{M}{(n+1)^2}\right] - \frac{2M}{(n+1)^2}\right\} + 1 =$$
$$= \frac{n+4}{(n+1)^4}M^2 - \frac{2(n+3)M}{(n+1)^2} + (n+1)^2 \equiv K(M).$$

Since $0 < M \leq M_n$, where M_n is the positive root of the equation $K(M) = M^2$, we deduce $L(m, \theta, z) \geq M^2$, which yields (7). Hence the subordination (6) holds and we obtain (3), which completes the proof of Theorem 1.

The following two theorems are integral versions of Theorem 1.

Theorem 2. *If $g \in A_n$ satisfies $|g'(z) - 1| < M_n$, where M_n is given by (2), then*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \ for\ z \in U,$$

where

$$f(z) = \int_0^z \frac{g(t)}{t} dt = \int_0^1 \frac{g(uz)}{u} du.$$

Theorem 3. *If $g \in A_n$ satisfies $|g'(z) - 1| < M_n$, where M_n is given by (2), then*

$$Re \int_0^1 \frac{g(uz)}{ug(z)} du > \frac{1}{2}, \ for\ z \in U.$$

4. EXAMPLES

Example 1. If we let $g(z) = (\sin \lambda z)/\lambda$, then $g \in A_2$ and we have

$$|g'(z) - 1| = 2 \left| \sin^2 \frac{\lambda z}{2} \right| \leq 2\mathrm{sh}^2 \frac{|\lambda z|}{2} < 2\mathrm{sh}^2 \frac{|\lambda|}{2} \leq M_2 = \frac{81}{10 + \sqrt{721}} = 2.198$$

for $z \in U$ and

$$|\lambda| \leq \ln \left[1 + M_2 + \sqrt{M_2(M_2 + 2)} \right] = 1,83\ldots$$

and by Theorem 3 we deduce

$$Re \frac{Si(z)}{\sin z} < \frac{1}{2}, \ \text{for } |z| < 1,83\ldots \tag{8}$$

where

$$Si(z) = \int_0^1 \frac{\sin uz}{u} du = \int_0^z \frac{\sin t}{t} dt.$$

The inequality (8) is an improvement of Example 1 in [4].

Example 2. If we let $g(z) = (\tan \lambda z)/\lambda$, then $g \in A_2$ and we have

$$|g'(z) - 1| = |\tan^2 \lambda z| \leq \tan^2 |\lambda z| < \tan^2 |\lambda| \leq M_2$$

for $z \in U$ and

$$|\lambda| \leq \arctan \sqrt{M_2} = 0.977\ldots$$

and by Theorem 3 we deduce

$$Re \int_0^1 \frac{\tan uz}{u \tan z} du > \frac{1}{2}, \ \text{for } |z| < 0.977\ldots \tag{9}$$

The inequality (9) is an improvement of Example 4 in [4].

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