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AN EXISTENCE RESULT ON AN UNBOUNDED REAL INTERVAL FOR AN INTEGRAL INCLUSION OF VOLTERRA TYPE

MOUFFAK BENCHOHRA, ABDELKADER BOUCHERIF and JUAN J. NIETO

Abstract. *In this note an extension of Schaefer's theorem to multivalued maps is used to investigate the existence of solutions on an unbounded real interval for an integral inclusion of Volterra type.*

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1. INTRODUCTION

In the past few years, several papers have been devoted to the study of integral equations on real compact intervals by different authors under different conditions on the kernel (see for instance [4], [5], [6], [7], [10], [8], [13], [14], [17] and the references therein). However very few results are available for integral inclusions on compact intervals see [1], [2], [5], [7] and [14].

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point arguments, nonlinear alternative, topological transversality or iterative methods.

In this note, we shall be concerned with the existence of solutions on an unbounded real interval for the integral inclusion:

$$y(t) \in \int_0^t K(t,s)F(s,y(s))ds + g(t) \quad \text{for} \quad t \in J := [0,\infty) \tag{1.1}$$

where $F : J \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a convex-multivalued map, $K : D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, and $g : J \longrightarrow \mathbb{R}^N$ a single-valued map.

We shall generalize to an unbounded real interval the problem (1.1) considered by O'Regan [15] on a compact interval. The method we are going to use is to reduce the existence of solutions to the integral inclusion (1.1) to the search for fixed points of a suitable multivalued map on the Fréchet space $C(J, \mathbb{R}^N)$ and we make use of an extension to multivalued maps on locally convex topological spaces due to Ma [12] of Schaefer's theorem [16].

2. PRELIMINARIES

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.
J_m is the compact real interval $[0, m] (m \in \mathbb{N})$.
$C(J, \mathbb{R}^N)$ is the linear metric Fréchet space of continuous functions from J into $\mathbb{R}^N$ with the metric (see [5])

$$d(y,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y-z\|_m}{1+\|y-z\|_m} \quad \text{for each} \quad y, z \in C(J, \mathbb{R}^N),$$

where

$$\|y\|_m := \sup\{|y(t)| : t \in J_m\}.$$

$L^1_{loc}(J, \mathbb{R}^N)$ denotes the Banach space of functions $y : J \longrightarrow \mathbb{R}^N$ Lebesgue integrable.
U_p denotes the neighbourhood of 0 in $C(J, \mathbb{R}^N)$ defined by

$$U_p := \{y \in C(J, \mathbb{R}^N) : \|y\|_m \leq p\}.$$

The convergence in $C(J, \mathbb{R}^N)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J, \mathbb{R}^N)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \longrightarrow 0$ in $C(J_m, \mathbb{R}^N)$ as $j \longrightarrow \infty$.
$M \subseteq C(J, \mathbb{R}^N)$ is a bounded set if and only if there exists a positive function $\phi \in C(J, \mathbb{R})$ such that

$$|y(t)| \leq \phi(t) \quad \text{for all} \quad t \in J \quad \text{and} \quad y \in M.$$

The Ascoli-Arzela theorem says that a set $M \subseteq C(J, \mathbb{R}^N)$ is compact if and only if for each $m \in \mathbb{N}$, M is a compact in the Banach space $(C(J_m, \mathbb{R}^N), \|.\|_m)$.
$L^\infty(J, \mathbb{R})$ is the space of essentially bounded measurable functions from J into $\mathbb{R}$.

Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \longrightarrow X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_0)$, there exists an open neighbourhood A of x_0 such that $G(A) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$, $y_n \in Gx_n$ imply $y_0 \in Gx_0$).
G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $CC(\mathbb{R}^N)$ denotes the set of all compact, convex and nonempty subsets of $\mathbb{R}^N$.

A multivalued map $G : J \longrightarrow CC(\mathbb{R}^N)$ is said to be measurable if for each $x \in \mathbb{R}^N$ the function $t \longmapsto d(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}$ is measurable. For more details on multivalued maps see [3] .

Let us list the following hypotheses:

(H1) $F : J \times \mathbb{R}^N \longrightarrow CC(\mathbb{R}^N); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in \mathbb{R}^N$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, \mathbb{R}^N)$ the set

$$S^1_{F,y} = \left\{ v \in L^1(J, \mathbb{R}^N) : \ v(t) \in F(t, y(t)) \text{ for a.e. } \ t \in J \right\}$$

is nonempty;

(H2) for each $t \in J_m$ $(m \in \{1, 2, ...\})$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J_m;

(H3) the map $t \longmapsto K_t$ is continuous from J_m to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H4) $g : J \longrightarrow \mathbb{R}^N$ is a continuous single- valued map;

(H5) there exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ with $\int_0^\infty \frac{du}{\psi(u)} = \infty$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$|F(t, y)| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|) \quad \text{for a.e. } \ t \in J \ \text{ and all } \ y \in \mathbb{R}^N;$$

By a solution to (1.1), we mean a function $y \in C(J, \mathbb{R}^N)$ that satisfies the integral inclusion (1.1) on J.

Remark 1 *If J is a compact real interval then for each $y \in C(J, \mathbb{R}^N)$ the set $S^1_{F,y}$ is nonempty (see* [11] *).*

The following lemmas are crucial in the proof of our main result (Theorem 3.1):

Lemma 1 [11] *Let I be a compact real interval. Let F be a multivalued map satisfying (H1) and let Γ be a linear continuous mapping from $L^1(I, \mathbb{R}^N)$ to $C(I, \mathbb{R}^N)$, then the operator*

$$\Gamma \circ S^1_F : C(I, \mathbb{R}^N) \longrightarrow CC(C(I, \mathbb{R}^N)), \ y \longmapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I, \mathbb{R}^N) \times C(I, \mathbb{R}^N)$.

Lemma 2 [9] *Let $[0, T]$ be a real compact interval. If $p \in L^1([0, T], \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is increasing with*

$$\int_0^\infty \frac{du}{\psi(u)} = \infty,$$

then the integral equation

$$z(t) = z_0 + \int_0^t p(s)\psi(z(s))ds, \quad \textit{for} \quad t \in [0, T],$$

has for each $z_0 \in \mathbb{R}_+$ *a unique solution* z. *If* $u \in C([0,T],\mathbb{R}^N)$ *satisfies the integral inequality*

$$|u(t)| \leq z_0 + \int_0^t p(s)\psi(|u(s)|)ds \quad for \quad t \in [0,T],$$

then $|u| \leq z$.

Lemma 3 [12] . *Let* X *be a locally convex space. Let* $G : X \longrightarrow X$ *be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighbourhood* U_p *of* 0 *for which* $G(U_p)$ *is a relatively compact set for each* $p \in \mathbb{N}$. *If the set*

$$M := \{y \in X : \lambda y \in G(y) \quad for\ some \quad \lambda > 1\}$$

is bounded, then G *has a fixed point.*

3. MAIN RESULT

Theorem 1 *Assume the hypotheses (H1), (H2), (H3), (H4) and (H5) are satisfied, then the integral inclusion (1.1) has at least one solution.*

Proof. A solution of (1.1) is a fixed point for the multivalued map $G : C(J,\mathbb{R}^N) \longrightarrow C(J,\mathbb{R}^N)$ defined by

$$G(y) := \left\{h \in C(J,\mathbb{R}^N) : h(t) = \int_0^t K(t,s)v(s)ds + g(t) \quad : v \in S^1_{F,y}\right\}$$

where

$$S^1_{F,y} = \left\{v \in L^1(J,\mathbb{R}^N) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in J\right\}.$$

We shall show that $G(U_p)$ is relatively compact for each $p \in \mathbb{N}$ and G is u.s.c. with convex closed values. The proof will be given in several steps

Step 1: $G(y)$ is convex for each $y \in C(J,\mathbb{R}^N)$.
Indeed, if h, $\overline{h}$ belong to $G(y)$ and $0 \leq k \leq 1$, then for each $t \in J$ we have

$$[kh + (1-k)\overline{h}](t) = \int_0^t K(t,s)[kv(s) + (1-k)\overline{v}(s)]ds + g(t).$$

Since $S^1_{F,y}$ is convex (because F has convex values) then

$$kh + (1-k)\overline{h} \in Gy.$$

Step 2: $G(U_p)$ is bounded in $C(J,\mathbb{R}^N)$ for each $p \in \mathbb{N}$.
Indeed, it is enough to show that there exists a positive constant c such that for each $h \in Gy, y \in U_p$ one has $|h| \leq c$.
If $h \in Gy$, then there exists $v \in S^1_{F,y}$ such that for each $t \in J_m$ we have

$$h(t) = \int_0^t K(t,s)v(s)ds + g(t)$$

Thus for each $t \in J_m$ we have

$$\begin{aligned} |h(t)| &\leq \int_0^t |K(t,s)||v(s)|ds + |g(t)| \\ &\leq \|p\|_{L^1} \sup_{t\in J_m} K(t) \sup_{t\in J_m} \psi(|y(t)|) + \sup_{t\in J_m} |g(t)| := c. \end{aligned}$$

Step 3: For each $p \in \mathbb{N}$, G sends $U_p \in C(J, \mathbb{R}^N)$ into equicontinuous set.
Let $t_1, t_2 \in J_m, t_1 < t_2$ and U_p be a neighbourhood of 0 in $C(J, \mathbb{R}^N)$ for $p \in \mathbb{N}$.
For each $y \in U_p$ and $h \in Gy$, we have

$$\begin{aligned} |h(t_2) - h(t_1)| &= \Big| \int_0^{t_2} K(t_2,s)v(s)ds - \int_0^{t_1} K(t_1,s)v(s)ds + g(t_2) - g(t_1)\Big| \\ &= \Big| \int_0^{t_1} [K(t_2,s) - K(t_1,s)]v(s)ds \\ &\quad + \int_{t_1}^{t_2} K(t_2,s)v(s)ds + g(t_2) - g(t_1)\Big|. \end{aligned}$$

From hypotheses (H2)-(H5) we get

$$\begin{aligned} |h(t_2) - h(t_1)| &\leq \|K(t_2,.) - K(t_1,.)\|_{L^\infty} \int_0^{t_m} |v(s)|ds \\ &\quad + \Big(\sup_{t\in J_m} K(t)\Big) \int_{t_1}^{t_2} |v(s)|ds + |g(t_2) - g(t_1)| \\ &\leq \|K(t_2,.) - K(t_1,.)\|_{L^\infty} \|p\|_{L^1} \sup_{t\in J_m} \psi(|y(t)|) \\ &\quad + \sup_{t\in J_m} K(t)(t_2 - t_1)\|p\|_{L^1} \sup_{t\in J_m} \psi(|y(t)|) + |g(t_2) - g(t_1)|. \end{aligned}$$

As a consequence of Step 2, Step 3 together with the Ascoli-Arzela theorem we can conclude that $G : C(J, \mathbb{R}^N) \longrightarrow C(J, \mathbb{R}^N)$ is a compact multivalued map.

Step 4: G has a closed graph.
Let $y_n \dashrightarrow y_0$, $h_n \in G(y_n)$, $h_n \longrightarrow h_0$. We shall prove that $h_0 \in G(y_0)$.
$h_n \in G(y_n)$ means that there exists $v_n \in S^1_{F,y_n}$ such that

$$h_n(t) = \int_0^t K(t,s)v_n(s)ds + g(t).$$

We must prove that there exists $v_0 \in S^1_{F,y_0}$ such that

$$h_0(t) = \int_0^t K(t,s)v_0(s)ds + g(t). \tag{3.1}$$

The idea is then to use the fact that
(i) $h_n \longrightarrow h_0$ as $n \longrightarrow \infty$;
(ii) $h_n - g \in \Gamma(S^1_{F,y_n})$ where

$$(\Gamma v)(t) := \int_0^t K(t,s)v(s)ds.$$

If $\Gamma \circ S_F^1$ is a closed graph operator, we would be done. But we don't know whether $\Gamma \circ S_F^1$ is a closed graph operator. So, we cut the functions y_n, h_n, v_n and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbb{N}$. Then, using Lemma 2.1, in this case we are able to affirm that (3.1) is true on the compact interval $[k, k+1]$, i.e.

$$h_0\Big|_{[k,k+1]}(t) = \int_0^t K(t,s)v_0^k(s)ds + g(t)$$

for a suitable L^1-selection v_0^k of $F(t, y_0(t))$ on the interval $[k, k+1]$.
At this point we can paste the functions v_0^k obtaining the selection v_0 defined by

$$v_0(t) = v_0^k(t) \quad \text{for} \quad t \in [k, k+1).$$

We obtain then that v_0 is a L^1-selection and (3.1) will satisfied.
We give now the details.
Clearly we have

$$\|(h_n - g) - (h_0 - g)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Now, we consider for all $k \in \mathbb{N}$, the mapping

$$S_F^{1,k} : C([k, k+1], \mathbb{R}^N) \longrightarrow L^1([k, k+1], \mathbb{R}^N)$$

$$u \longmapsto S_{F,u}^{1,k} := \{f \in L^1([k, k+1], \mathbb{R}^N) : f(t) \in F(t, u(t)) \quad \text{for a.e.} \quad t \in [k, k+1]\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k, k+1], \mathbb{R}^N) \longrightarrow C([k, k+1], \mathbb{R}^N)$$

$$v \longmapsto \Gamma_k(v)(t) = \int_0^t K(t,s)v(s)ds.$$

From Lemma 2.1, it follows that $\Gamma_k \circ S_F^{1,k}$ is a closed graph operator for all $k \in \mathbb{N}$.
Moreover, we have that

$$(h_n - g)\Big|_{[k,k+1]}(t) \in \Gamma_k(S_{F,y_n}^{1,k}).$$

This, besides to $y_n \longrightarrow y_0$ and Lemma 2.1, furnishs

$$h_0\Big|_{[k,k+1]}(t) = \int_0^t K(t,s)v_0^k(s)ds + g(t)$$

for some $v_0^k \in S_{F,y_0}^{1,k}$. So the function v_0 defined on J by

$$v_0(t) = v_0^k(t) \quad \text{for} \quad t \in [k, k+1)$$

is in S_{F,y_0}^1 since $v_0(t) \in F(t, y_0(t))$ for a.e. $t \in J$.

It remains now to prove that the set

$$M := \{y \in C(J, \mathbb{R}^N) : \lambda y \in G(y), \ \lambda > 1\}$$

is bounded to conclude (by Lemma 2.3) that G has fixed points.
For this, let $\lambda y \in G(y)$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}^1$ such that

$$y(t) = \lambda^{-1} \int_0^t K(t,s)v(s)ds + \lambda^{-1}g(t) \quad \text{for all} \quad t \in J_m.$$

In view of (H2), (H3), (H4) and (H5) we have for each $t \in J_m$

$$\begin{aligned} |y(t)| &\leq \int_0^t |K(t,s)|p(s)\psi(|y(s)|)ds + \|g\|_m \\ &\leq \sup_{t\in J_m} K(t) \int_0^t p(s)\psi(|y(s)|)ds + \|g\|_m. \end{aligned}$$

As a consequence of Lemma 2.2, we obtain

$$\|y\|_m \leq \|z\|_m,$$

where z is the unique solution on J_m of the integral equation

$$z(t) - \|g\|_m = \sup_{t\in J_m} K(t) \int_0^t p(s)\psi(z(s))ds. \tag{3.2}$$

So M is bounded.
Set $X := C(J, \mathbb{R}^N)$. As a consequence of Lemma 2.3 we can conclude that the multivalued map G has a fixed point y which is a solution to (1.1). □

Remark 2 *The condition on ψ in hypothesis (H5) with Lemma 2.2 imply the existence and the uniqueness of the solution to (3.2).*

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