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AN ASYMPTOTIC EXPANSION FOR SOME SECOND ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. *An asymptotic expansion (series) for the solution* u_ε *of the boundary value problem* (A_ε) *below is constructed. The n-th partial sum of this series is an asymptotic approximation for* u_ε *in* $[0,T]$ *to within accuracy of the order* $\varepsilon^{n+1/2}$. *In particular,* u_ε *approaches the solution u of the first order problem* (A_0), *in the entire interval* $[0,T]$, *if* $b = u(T)$ *and in* $[0,\delta]$, *for some positive* δ, *if* $b \neq u(T)$.

1. INTRODUCTION

In this paper, we intend to compare the solution of the boundary value problem:

$$\begin{cases} \varepsilon u_\varepsilon'' - u_\varepsilon' = Au_\varepsilon + f \ \text{ a.e. on } [0,T] \\ u_\varepsilon(0) = a, \ \ u_\varepsilon(T) = b, \end{cases} \tag{A_ε}$$

with the solution of the problem

$$\begin{cases} u' + Au = -f, \ \text{ a.e. on } [0,T] \\ u(0) = a. \end{cases} \tag{A_0}$$

Here ε is a small positive parameter, A is a linear operator on a Hilbert space H, f is a given function and $a, b \in D(A)$ are given. If in (A_ε) we take $\varepsilon = 0$, we obtain the problem (A_0), so (A_ε) is called the perturbed problem and (A_0) is the unperturbed problem.

Our main goal is to show that the difference $u_\varepsilon - u$ (in a suitably chosen measure) approximates zero on a subset $[0,\delta]$, for some δ, with $0 < \delta < T$. Moreover, since $u(t)$ does not approximate $u_\varepsilon(t)$ for small ε in the so - called boundary layer $(\delta, T]$, we shall construct an asymptotic expasion (or asymptotic series) for $u_\varepsilon(t)$, that is valid in the entire interval $[0,T]$.

First, let us establish the hypotheses under which we work and recall some existence results concerning the solutions of (A_ε) and (A_0).

We are given a real Hilbert space H (with the norm $\|\cdot\|$) and an operator $A : H \to H$ such that
(I1) A is a single-valued, linear, densely defined, symmetric and positive definite operator, i.e., there is a positive number ω, such that

$$(Ax, x) \geq \omega\|x\|^2, \ \ (\forall) x \in H. \tag{1.1}$$

(I2) $a, b \in D(A)$ (1.2)
(I3) $f \in W^{1,2}(0,T;H)$, i.e. $f, f' \in L^2(0,T;H)$. (1.3)

We know that, if A is maximal monotone set in $H\times H$, $a\in D(A)$ and $f\in W^{1,1}(0,T;H)$, then (A_0) has a unique strong solution u, which belongs to $W^{1,\infty}(0,T;H)$ ([6], [8], [9], [10]), i.e. $u(t)$ is absolutely continuous on $[0,T]$ and $du/dt\in L^{\infty}(0,T;H)$.

The problem $A\epsilon$ is a particular case of the more general boundary value problem

$$\begin{cases} pu''+ru'\in Au+f, & \text{a.e. on } [0,T] \\ u'(0)\in\alpha(u(0)-a), & u'(T)\in-\beta(u(T)-b), \end{cases} \tag{1.4}$$

where A,α,β are maximal monotone operators in a real Hilbert space H, with $0\in D(A)\cap D(\alpha)\cap D(\beta)$, $0\in\alpha(0)$, $0\in\beta(0)$, $a,b\in D(A)$, $f\in L^2(0,T;H)$ and $p,r:[0,T]\to\mathbb{R}$, $p,r\in W^{1,\infty}(0,T)$. One also assumes that, if A_λ is the Yosida approximation of A, i.e. $A_\lambda x=(1/\lambda)(x-(I+\lambda A)^{-1}x)$, then

$$\begin{gathered}(A_\lambda x-A_\lambda y,z)\geq 0,\ \ (\forall)z\in\alpha(x-y),\ \ \text{where } x-y\in D(\alpha),\\ (A_\lambda x-A_\lambda y,w)\leq 0,\ \ (\forall)w\in-\beta(x-y),\ \ \text{where } x-y\in D(\beta).\end{gathered}$$

The existence of the solutions of (1.4) was studied by L. Véron [12] and by R. Aftabizadeh and N.H. Pavel [2], [3]. One of these results ([2]) establishes that, if A is maximal strongly monotone operator in H, then (1.4) has a unique solution $u\in W^{2,2}(0,T;H)$.

Similar results for second order differential equations on half-axis, were obtained by N. Apreutesei [4], [5]. We recall one of these theorems, which we need later. Consider the problem

$$\begin{cases} pu''+ru'\in Bu+g, & \text{a.e. on } [0,\infty) \\ u'(0)\in\alpha(u(0)-a), \end{cases} \tag{1.5}$$

where B and α are maximal monotone operators in H, with $0\in D(B)\cap D(\alpha)$, $0\in\alpha(0)$, $a\in D(B)$ and $p,r\in W^{1,\infty}(0,\infty)$, $p(t)\geq c>0$, $r(t)\geq r_0>0$, $(\forall)t\in[0,\infty)$. Let $\tilde r$ be the function given by

$$\tilde r(t)=\exp\left(\int_0^t (r(s)/p(s))ds\right) \tag{1.6}$$

and $L^2_{\tilde r}(0,\infty;H)$ be the space $L^2(0,\infty;H)$ with the weight function $\tilde r$, i.e. the inner product in $L^2_{\tilde r}(0,\infty;H)$ is

$$\ll u,v\gg=\int_0^\infty \tilde r(t)(u(t),v(t))dt,\ \ (\forall)u,v\in L^2_{\tilde r}(0,\infty;H). \tag{1.7}$$

THEOREM 1.1. *If the above hypotheses hold, $D(\alpha)$ is bounded in H, $f\in L^2_{\tilde r}(0,\infty;H)$ and*

$$(A_\lambda x-A_\lambda y,z)\geq 0,\ \ \forall z\in\alpha(x-y),\ \ \textit{where } x-y\in D(\alpha), \tag{1.8}$$

then (1.5) has at least one solution $u\in H^2_{\tilde r}(0,\infty;H)=\{u/u,u',u''\in L^2_{\tilde r}(0,\infty;H)\}$.

We come back to (1.4) and put $\alpha=\beta=\partial j$, where $j:H\to(-\infty,+\infty]$,

$$j(x)=\begin{cases} 0, & x=0 \\ +\infty, & \text{otherwise.}\end{cases} \tag{1.9}$$

Then $D(\partial j)=\{0\}$ and $\partial j(0)=H$, so the boundary conditions become $u(0)=a$, $u(T)=b$, as in $(A\varepsilon)$. Taking $p(t)=\varepsilon$, $r(t)=-1$, $(\forall)t\in[0,T]$, we obtain the problem (A_ε).

Therefore, under the hypotheses (I1)-(I3), the problem $(A\varepsilon)$ has a unique solution $u_\varepsilon\in W^{2,2}(0,T;H)$. Moreover, as we shall prove in section 2, if $f\in W^{1,2}(0,T;H)$, then $u_\varepsilon\in W^{2,\infty}(0,T;H)$ (see Theorem 2.1).

The motivation of this paper is:

1. First of all, the elliptic equation from (1.4) is too abstract. If we particularise $p \equiv \varepsilon$, $r \equiv -1$ and A, α, β as above, we obtain the problem $(A\varepsilon)$. It's solution, u_ε, converges to the solution u of (A_0) and the problem (A_0) is the model of many physical phenomenon. Therefore, (A_ε) can be interpreted as an approximation (in a certain sense) of the mathematical models of these processes or phenomenon.
2. On the other hand, we shall prove that, under the assumptions (I1)-(I3), u_ε is in $W^{2,\infty}(0,T;H)$, so the solution $u \in W^{1,\infty}(0,T;H)$ of (A_0) is approximated by the smoother function u_ε. This justifies the necessity of the elliptic regularization.
3. We show that, if $b = u(T)$, then $u_\varepsilon - u \to 0$, as $\varepsilon \to 0$, in $C(0,T;H)$ and, moreover this difference is of order ε to a positive power; if $b \neq u(T)$, then u_ε approximates u just outside of a neighbourhood $(\delta, T]$ of T.
4. Moreover, we construct an approximation with higher accuracy and the complete asymptotic expansion for the solution of the problem (A_ε), in the spirit of A. Vasilieva, V. Butuzov and L. Kalachev [11].

Other papers which deal with the approximation of the solutions of an unperturbed problem by the solutions of a perturbed problem are [1], [7].

The structure of this work is the following. In the second section, we give a regularization result for the problem (A_ε). The next paragraph is concerned with the construction of the asymptotic expansion (series) of the solution u_ε of the perturbed problem (A_ε). In section 4, we justify the asymptotic expansion and we estimate the remainder term. In the last paragraph, an application of this theory to a partial differential equation is given.

2. A REGULARITY RESULT

As we mention above, the solution u_ε of (A_ε) is in $W^{2,2}(0,T;H)$. We shall establish that, in fact, u_ε belongs to $W^{2,\infty}(0,T;H)$, which explain why we approximate the solution $u \in W^{1,\infty}(0,T;H)$ of (A_0) by the solution u_ε of (A_ε). More exactly, we give

THEOREM 2.1. *If $f \in W^{1,2}(0,T;H)$ and A is a linear operator satisfying condition (I1), the solution u_ε of (A_ε) is in $W^{2,\infty}(0,T;H)$ and, for every $\varepsilon > 0$, we have the estimate*

$$\begin{aligned}\frac{\varepsilon}{2}\|u_\varepsilon''(t)\|^2 \le \frac{1}{\varepsilon}\|u_\varepsilon'(0)\|^2 + \frac{2}{\varepsilon}\|Aa\|^2 + \frac{2}{\varepsilon}\|f(0)\| + 2\int_0^t \|u_\varepsilon''(s)\|^2 ds + \\ + \int_0^t \|f'(s)\|^2 ds + \frac{1}{2}\sup_{t\in[0,T]}(Au_\varepsilon'(t), u_\varepsilon'(t)),\quad t \in [0,T].\end{aligned} \tag{2.1}$$

Proof. We know that, for every fixed positive ε, $u_\varepsilon \in W^{2,2}(0,T;H)$, so $u_\varepsilon, u_\varepsilon' \in L^\infty(0,T;H)$. Without loss of generality, suppose that A is a bounded operator. If not, we replace A by A_λ, we denote by $u_{\varepsilon\lambda}$ the solution of the corresponding approximation problem and we take the limit for $\lambda \downarrow 0$, taking into account the fact that $u_{\varepsilon\lambda} \to u_\varepsilon$, $u_{\varepsilon\lambda}' \to u_\varepsilon'$, $u_{\varepsilon\lambda}'' \to u_\varepsilon''$ and $A_\lambda u_{\varepsilon\lambda} \to Au_\varepsilon$; thus we obtain (2.1). Let us show now that $u_\varepsilon'' \in L^\infty(0,T;H)$ and (2.1).

To do this, we write the equation in $t+h$ and in t, we subtract them and we multiply this difference by $u'_\varepsilon(t+h) - u'_\varepsilon(t)$. One obtains

$$\begin{aligned} &\varepsilon(u''_\varepsilon(t+h) - u''_\varepsilon(t),\ u'_\varepsilon(t+h) - u'_\varepsilon(t)) - \|u'_\varepsilon(t+h) - u'_\varepsilon(t)\|^2 = \\ &\qquad (Au_\varepsilon(t+h) - Au_\varepsilon(t),\ u'_\varepsilon(t+h) - u'_\varepsilon(t)) + \\ &\quad +(f(t+h) - f(t),\ u'_\varepsilon(t+h) - u'_\varepsilon(t)), \text{ a.e. on } (0, T-h). \end{aligned} \tag{2.2}$$

One integrates (2.2) on $[0,t]$ (with $0 \le t \le T$), one uses the linearity of A and the fact that, for $v(t) = u_\varepsilon(t+h) - u_\varepsilon(t)$,

$$\int_0^t (Av, v')ds = \frac{1}{2}(Av(t), v(t)) - \frac{1}{2}(Av(0), v(0)). \tag{2.3}$$

We deduce that

$$\begin{aligned} &\frac{\varepsilon}{2}\|u'_\varepsilon(t+h) - u'_\varepsilon(t)\|^2 - \frac{\varepsilon}{2}\|u'_\varepsilon(h) - u'_\varepsilon(0)\|^2 - \int_0^t \|u'_\varepsilon(s+h) - u'_\varepsilon(s)\|^2 ds \le \\ &\qquad \le \frac{1}{2}(A(u_\varepsilon(t+h) - u_\varepsilon(t)),\ u_\varepsilon(t+h) - u_\varepsilon(t)) + \\ &\quad + \int_0^t (f(s+h) - f(s),\ u'_\varepsilon(s+h) - u'_\varepsilon(s))ds, \text{ a.e. on } (0, T-h). \end{aligned} \tag{2.4}$$

Let us estimate now $\|u'_\varepsilon(h) - u'_\varepsilon(0)\|^2$. Integrating the given equation on $[0,h]$, we find

$$\varepsilon\|u'_\varepsilon(h) - u'_\varepsilon(0)\| \le \|u_\varepsilon(h) - u_\varepsilon(0)\| + \int_0^h \|Au_\varepsilon(s)\|ds + \int_0^h \|f(s)\|ds,$$

so

$$\frac{\varepsilon}{2}\|u'_\varepsilon(h) - u'_\varepsilon(0)\|^2 \le \frac{1}{\varepsilon}\|u_\varepsilon(h) - u_\varepsilon(0)\|^2 + \frac{2}{\varepsilon}\left(\int_0^h \|Au_\varepsilon(s)\|ds\right)^2 + \frac{2}{\varepsilon}\left(\int_0^h \|f(s)\|ds\right)^2. \tag{2.5}$$

The inequalities (2.4) and (2.5) imply

$$\begin{aligned} &\frac{\varepsilon}{2}\|u'_\varepsilon(t+h) - u'_\varepsilon(t)\|^2 \le \frac{1}{\varepsilon}\|u_\varepsilon(h) - u_\varepsilon(0)\|^2 + \frac{2}{\varepsilon}\left(\int_0^h \|Au_\varepsilon(s)\|ds\right)^2 + \frac{2}{\varepsilon}\left(\int_0^h \|f(s)\|ds\right)^2 + \\ &\qquad + \int_0^t \|u'_\varepsilon(s+h) - u'_\varepsilon(s)\|^2 ds + \frac{1}{2}(A(u_\varepsilon(t+h) - u_\varepsilon(t)), u_\varepsilon(t+h) - u_\varepsilon(t)) + \\ &\qquad\qquad + \int_0^t (f(s+h) - f(s), u'_\varepsilon(s+h) - u'_\varepsilon(s))ds. \end{aligned} \tag{2.6}$$

We divide by h^2 and let $h \to 0$. In view of

$$\lim_{h\to 0}\frac{1}{h^2}\left(\int_0^h \|Au_\varepsilon(s)\|ds\right)^2 = \|Au_\varepsilon(0)\|^2$$

and

$$\lim_{h\to 0}\frac{1}{h^2}\left(\int_0^h \|f(s)\|ds\right)^2 = \|f(0)\|^2,$$

we see that (2.1) holds, so $u''_\varepsilon \in L^\infty(0,T;H)$.

3. THE CONSTRUCTION OF THE ASYMPTOTIC EXPANSION OF u_ε

Let us give now some formal definitions. Consider the equations

$$L_0 u = f_0, \tag{P_0}$$

$$L_0 u_\varepsilon + \varepsilon L_1 u_\varepsilon = f_0 + \varepsilon f_1, \tag{P_ε}$$

where L_0, L_1 are two given operators, f_0, f_1 are known functions, $\varepsilon > 0$ is a small parameter and $u(x), u_\varepsilon(x), x \in D$ are the solutions of these problems, satisfying some initial or boundary conditions.

DEFINITION 3.1. We say that (P_ε) is regularly perturbed in the domain D if

$$\sup_{x \in D} \|u_\varepsilon(x) - u_0(x)\| \to 0, \ \text{ as } \varepsilon \to 0. \tag{3.1}$$

Otherwise, (P_ε) is called singularly perturbed problem.

Remark 3.1. We see that, in the case of the regularly perturbed problem, the solutions u and u_ε of (P_0), (P_ε) are close to each other and in the case of singular perturbations, u will not be in the proximity of (u_ε), at least on a subset D_0 of D. This subset D_0 is called a boundary layer.

Let D_1 be a subset of D and $U_\varepsilon(x)$ a function which is defined on D_1. In particular, D_1 may coincide with D.

DEFINITION 3.2. The function $U_\varepsilon(x)$ is said to be an asymptotic approximation for the solution u_ε of (P_ε), with respect to the parameter ε, in the subdomain D_1, if

$$\sup_{x \in D_1} \|u_\varepsilon(x) - U_\varepsilon(x)\| \to 0, \ \text{ as } \ \varepsilon \to 0. \tag{3.2}$$

Furthermore, if

$$\sup_{x \in D_1} \|u_\varepsilon(x) - U_\varepsilon(x)\| = O(\varepsilon^k), \tag{3.2$'$}$$

then we say that $U_\varepsilon(x)$ is an asymptotic approximation for $u_\varepsilon(x)$ in D_1, to within accuracy of the order ε^k.

The symbol $\alpha(\varepsilon) = O(\varepsilon^k)$ means that, there exists $c, \ \varepsilon_0 > 0$, such that,

$$\|\alpha(\varepsilon)\| \leq c\varepsilon^k, \ \text{ for all } \ 0 \leq \varepsilon \leq \varepsilon_0.$$

In this section, we shall present a method for the construction of an asymptotic approximation for the solution u_ε of (A_ε), in entire interval $[0, T]$. Namely, we shall consider the series

$$\sum_{k=0}^{\infty} \varepsilon^k u_k(t), \tag{3.3}$$

where $u_k \in L^2(0, T; H)$. The n-th partial sum

$$U_{n\varepsilon}(t) = \sum_{k=0}^{n} \varepsilon^k u_k(t), \tag{3.4}$$

will give us the asymptotic approximation of u_ε with an accuracy of order $\varepsilon^{n+1/2}$, as we shall see in section 4.

DEFINITION 3.3. The series (3.3) which satisfies the above condition is called the asymptotic expansion (asymptotic series) for u_ε in the domain D, as $\varepsilon \to 0$.

When we refer to the asymptotic series, we often mean an algorithm which allows us to find the terms of the asymptotic series, until a certain order n. We say that the asymptotic series is constructed when such an algorithm is presented.

Now, we come back to our problems (A_0) and (A_ε). It is natural to ask whether $u(t)$ is the asymptotic approximation for $u_\varepsilon(t)$ as $\varepsilon \to 0$. Observe that $u(t)$ will not in general satisfy the boundary condition $u(T) = b$, therefore the problem (A_ε) is singularly perturbed, i.e. at least in some vicinity of the final point $t = T$, the function $u(t)$ will not be close to $u_\varepsilon(t)$. This vicinity $(\delta, T]$ is said to be the boundary layer. But does $u(t)$ approximate $u_\varepsilon(t)$ outside this vicinity? We wish to construct an asymptotic approximation for $u_\varepsilon(t)$, which is valid in the whole interval $[0, T]$ and we also want to find its order of accuracy.

To this end, we are looking for an asymptotic expansion for the solution u_ε of (A_ε), in a form that is quite typical for singularly perturbed problems:

$$u_\varepsilon(t) = \overline{u}_\varepsilon(t) + \Pi(\tau), \quad \text{with} \quad \tau = (T - t)/\varepsilon, \tag{3.5}$$

where $\overline{u}_\varepsilon(t)$ is the regular power series and $\Pi(\tau)$ is a boundary layer series in the neighbourhood $(\delta, T]$ of $t = T$, i.e.

$$\overline{u}_\varepsilon(t) = u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^k u_k(t) + \ldots, \tag{3.6}$$

$$\Pi(\tau) = \Pi_0(\tau) + \varepsilon \Pi_1(\tau) + \ldots + \varepsilon^k \Pi_k(\tau) + \ldots \tag{3.7}$$

So, the asymptotic expansion has the form

$$u_\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k (u_k(t) + \Pi_k(\tau)). \tag{3.8}$$

Here, $\Pi_k(\tau)$ should be boundary functions, which means that they approach zero as $\tau \to \infty$. Hence, we assume a priori that

$$\Pi_k(\infty) = 0, \quad n = 0, 1, 2, \ldots \tag{3.9}$$

The functions $\Pi_k(\tau)$ are important only in a neighbourhood of $t = T$ (i.e. $\tau = 0$), namely in the boundary layer $(\delta, T]$.

Using (3.5) in the differential equation from (A_ε), it follows via the linearity of A

$$\varepsilon \overline{u}_\varepsilon''(t) + \frac{1}{\varepsilon^2}\Pi''(\tau) - \overline{u}_\varepsilon'(t) + \frac{1}{\varepsilon}\Pi'(\tau) = A\overline{u}_\varepsilon(t) + A\Pi(\tau) + f(t), \quad \text{a.e. on } [0, T]. \tag{3.10}$$

Substituting the series (3.6) and (3.7) into (3.10) and equating the coefficients of like powers of ε, separately for the terms depending on t and on τ, respectively, we infer

$$\begin{gathered} u_0'(t) + Au_0(t) = -f(t) \ \text{ a.e. } t \in [0, T] \\ u_1'(t) + Au_1(t) = u_0''(t) \ \text{ a.e. } t \in [0, T] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ u_k'(t) + Au_k(t) = u_{k-1}''(t) \ \text{ a.e. } t \in [0, T] \end{gathered}$$

and

$$\begin{gathered} \Pi_0''(\tau) + \Pi_0'(\tau) = 0 \ \text{ a.e. } \tau \in [0, \infty) \\ \Pi_1''(\tau) + \Pi_1'(\tau) = A\Pi_0(\tau) \ \text{ a.e. } \tau \in [0, \infty) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Pi_k''(\tau) + \Pi_k'(\tau) = A\Pi_{k-1}(\tau) \ \text{ a.e. } \tau \in [0, \infty). \end{gathered}$$

Now, let us substitute the expansion (3.8) into boundary conditions of (A_ε) and take into account (3.9), we obtain

$$u_0(0)=a,\ \ u_k(0)=0,\ \ n\geq 1,$$
$$u_0(T)+\Pi_0(0)=b,\ \ u_k(T)+\Pi_k(0)=0,\ \ k\geq 1.$$

If this equalities hold, then the series (3.8) satisfies the boundary condition $u_\varepsilon(T)=b$, hence we can say that the role of the series (3.7) is to satisfy, together with the series (3.6), the imposed boundary condition $u_\varepsilon(T)=b$. Thus, the Π-functions compensate for the discrepancy introduced by the regular part of the asymptotic expansion to the boundary condition $u_\varepsilon(T)=b$.

Unless (3.9), we need to require on the boundary functions the condition

$$\Pi_k\in H^2(0,\infty;H),\ \ k\geq 0, \tag{3.11}$$

i.e. $\Pi_k,\Pi_k',\Pi_k''\in L^2(0,\infty;H)$. It is known that, if $P\in H^2(0,\infty;H)$, then

$$\lim_{t\to\infty}\|P(t)\|=0,\ \ \lim_{t\to\infty}\|P'(t)\|=0,$$

so (3.11) is stronger then (3.9).

Now, we can write the problems for the terms of the series (3.8). For the leading term $u_0(t)$ of the regular part of the asymptotic expansion, we obtain the boundary value problem

$$\begin{cases} u_0'(t)+Au_0(t)=-f(t), \ \text{a.e. on } [0,T] \\ u_0(0)=a, \end{cases} \tag{3.12}$$

which evidently coincides with the unperturbed problem (A_0).

For $u_k(t)$, $k\geq 1$, we deduce a problem of the same form with (3.12):

$$\begin{cases} u_k'(t)+Au_k(t)=u_{k-1}''(t) \ \text{a.e. on } [0,T] \\ u_k(0)=0. \end{cases} \tag{3.13}$$

In the zeroth order approximation of the boundary layer series, we have

$$\begin{cases} \Pi_0''(\tau)+\Pi_0'(\tau)=0 \ \text{a.e. on } [0,\infty) \\ \Pi_0(0)=b-u_0(T),\ \ \Pi_0\in H^2(0,\infty;H) \end{cases} \tag{3.14}$$

and for the higher-order terms $\Pi_k(\tau)$, $k\geq 1$, of the series (3.7), we arrive at

$$\begin{cases} \Pi_k''(\tau)+\Pi_k'(\tau)=A\Pi_{k-1}(\tau) \ \text{a.e. on } [0,\infty) \\ \Pi_k(0)=-u_k(T),\ \ \Pi_k\in H^2(0,\infty;H). \end{cases} \tag{3.15}$$

Replace the hypothesis (I3) by the more restrictive condition

$(I3)'$ *forsomefixed* $\quad f\in W^{n,2}(0,T;H)$ i.e. $f,f',\ldots,f^{(n)}\in L^2(0,T;H)$.

n ≥ 1.

$(I4)$ $\alpha_k=-f^{(k-1)}(0)+Af^{(k-2)}(0)-\ldots+(-1)^kA^{k-1}f(0)+(-1)^kA^ka\in D(A),\ \ k=\overline{1,n}.$

Let us show that each of these problems has a unique solution (under the assumptions (I1), (I2), (I3)′, (I4)).

First, (3.12) has a unique solution $u_0=u\in W^{1,\infty}(0,T;H)$, as we explained in Introduction.

Remark 3.2. Let $w\in W^{1,2}(0,T;H)$ be the solution of

$$\begin{cases} w'+Aw=-f^{(n)}, \ \text{a.e. } [0,T] \\ w(0)=\alpha_n. \end{cases}$$

Using the constants variation formula, we can easy see that $u^{(n)} = w$, so $u \in W^{n+1,2}(0,T;H)$.

Let us show now the uniqueness of the solution for (3.14) and (3.15).

PROPOSITION 3.1. *If (3.14) (or (3.15)) has solution $\Pi_k \in H^2(0,\infty;H)$, then it is unique.*

Proof. Let $\Pi_k, Q_k \in H^2(0,\infty;H)$ be two solutions of (3.14) (or (3.15)). Denoting by P the difference $P = P_k = \Pi_k - Q_k$, we observe that P verifies the problem

$$\begin{cases} P''(\tau) + P'(\tau) = 0, \ \text{a.e. on } [0,\infty) \\ P(0) = 0, \quad P \in H^2(0,\infty;H). \end{cases} \tag{3.16}$$

Let τ be a given number in $[0,\infty)$. Multiplying (3.16) by P and integrating over $[\tau,\infty)$, we have

$$\int_\tau^\infty [(P'',P) + (P',P)]ds = 0$$

or, equivalently

$$(P',P)\mid_\tau^\infty + \frac{1}{2}\|P\|^2 \mid_\tau^\infty = \int_\tau^\infty \|P'\|^2 ds. \tag{3.17}$$

Using the fact that $(P',P)(\infty) = 0$ and $\|P(\infty)\| = 0$, it follows that

$$(P'(\tau),P(\tau)) = -\frac{1}{2}\|P(\tau)\|^2 - \int_\tau^\infty \|P'\|^2 ds \le 0, \quad \tau \ge 0 \tag{3.18}$$

so, the function $\tau \to (1/2)\|P(\tau)\|^2$ is nonincreasing on $[0,\infty)$. This implies that $P(\tau) = 0$, for every $\tau \in [0,\infty)$, hence we have the uniqueness.

Now, we want to solve the problem (3.14). To do this, we shall prove that there is $x \in H$ such that $\Pi_0(\tau) = \phi(\tau)x$ verifies (3.14), where $\phi : [0,\infty) \to \mathbb{R}$ is the solution of the boundary value problem

$$\begin{cases} \phi''(\tau) + \phi'(\tau) = 0, \quad \tau \in [0,\infty) \\ \phi(0) = 1, \quad \phi(\infty) = 0. \end{cases} \tag{3.19}$$

The solution of (3.19) is $\phi(\tau) = e^{-\tau}$, so $\Pi_0(\tau) = e^{-\tau}x$. From $\Pi_0(0) = b - u_0(T)$, we find

$$\Pi_0(\tau) = (b - u_0(T))e^{-\tau}, \quad \tau \in [0,\infty). \tag{3.20}$$

Remark that $\|\Pi_0(\tau)\| \le ce^{-\tau}$, $(\forall)\tau \ge 0$, where $c = \|b - u_0(T)\| \ge 0$. The condition $\Pi_0 \in H^2(0,\infty;H)$ is also satisfied, because

$$\mid \Pi_0 \mid^2_{L^2(0,\infty;H)} = \mid \Pi_0' \mid^2_{L^2(0,\infty;H)} = \mid \Pi_0'' \mid^2_{L^2(0,\infty;H)} = \frac{1}{2}\|b - u_0(T)\|^2 < \infty.$$

Therefore, we can state

PROPOSITION 3.2. *The unique solution Π_0 of (3.14) is given by (3.20).*

Remark 3.3. Similarly, we can find the unique solution of (3.15) for $k = 1$, namely,

$$\Pi_1(\tau) = -e^{-\tau}u_1(T) - \tau e^{-\tau}A(b - u_0(T)). \tag{3.21}$$

For an arbitrary $k \in \mathbb{N}$, we cannot find the exact solution of (3.15), but we can apply, for example, Theorem 1.1 to deduce that (3.15) has a solution $\Pi_k \in H^2(0,\infty;H)$. Indeed, we put $\tilde{r}(\tau) = e^\tau$, via (1.6), $B = 0$ (so $B_\lambda = 0$), α as in (1.9), $p = r = 1$ and $g_k = A\Pi_{k-1}$. In view of the mathematical induction method, if $\Pi_{k-1} \in L^2_{\tilde{r}}(0,\infty;H)$, it follows that $g_k \in L^2_{\tilde{r}}(0,\infty;H)$ and, according to Theorem 1.1, we find $\Pi_k \in H^2_{\tilde{r}}(0,\infty;H) \subseteq H^2(0,\infty;H)$.

So, we deduced that the problems (3.12) - (3.15) have unique solutions $u_k \in W^{2,2}(0,T;H)$, $\Pi_k \in H^2(0,\infty;H)$, $k = \overline{0,n}$. Thus, the algorithm described allows one to define the terms of the series (3.6) and (3.7) up to the order n, inclusive (under the hypotheses (I1), (I2), (I3)$'$).

Remark 3.4. If $f \in W^{1,2}(0,T;H)$ (i.e. we have (I3), not (I3)$'$), we can find only the zeroth order terms of the series, u_0 and Π_0.

4. THE JUSTIFICATION OF THE ASYMPTOTIC EXPANSION

Let us denote by $S_{n\varepsilon}$ the n-th partial sum of the series (3.8), and by r_ε the remainder term, i.e.

$$S_{n\varepsilon}(t) = \sum_{k=0}^{n} \varepsilon^k [u_k(t) + \Pi_k(\tau)], \tag{4.1}$$

$$r_\varepsilon(t) = u_\varepsilon(t) - S_{n\varepsilon}(t). \tag{4.2}$$

We desire to estimate $\sup\limits_{0 \le t \le T} \|r_\varepsilon(t)\|$. The remainder term verifies the boundary value problem (which has a similar form to (A_ε)):

$$\begin{cases} \varepsilon r_\varepsilon'' - r_\varepsilon' = Ar_\varepsilon + \varepsilon^n (A\Pi_n(\tau) - \varepsilon u_n'') \text{ a.e. on } [0,T] \\ r_\varepsilon(0) = r_\varepsilon(T) = 0. \end{cases} \tag{4.3}$$

Remark 4.1. First, observe that, for a given $n \in I\!N$, $u_n'' \in L^2(0,T;H)$, via Remark 3.2. Then, the solution Π_n of (3.15) or (3.14) is in $L^2(0,\infty;H)$. Moreover, since $\tau = (T-t)/\varepsilon$, then

$$\int_0^T \|A\Pi_n(\tau)\|^2 dt = \varepsilon^2 \int_0^\infty \|A\Pi_n(\tau)\|^2 d\tau < \infty.$$

Then, there exists $M_n > 0$ such that

$$\int_0^T \left\|A\Pi_n\left(\frac{T-t}{\varepsilon}\right) - \varepsilon u_n''(t)\right\|^2 dt \le \varepsilon^2 M_n^2, \tag{4.4}$$

i.e. $A\Pi_n(\tau) - \varepsilon u_n''(t) \in L^2(0,T;H)$. Therefore, the problem (4.3) has a unique solution $r_\varepsilon \in W^{2,2}(0,T;H)$.

Now we can state the main result of this section, which will be the answer of the questions from Introduction.

THEOREM 4.1. *Under the hypotheses (I1), (I2), (I3)$'$, (I4), for sufficiently small ε, the series (3.8) is the asymptotic expansion (series) of the solution u_ε of (A_ε) in the interval $[0,T]$, as $\varepsilon \to 0$, i.e. the following estimate holds*

$$\sup_{0 \le t \le T} \|u_\varepsilon(t) - S_{n\varepsilon}(t)\| = 0(\varepsilon^{n+1/2}). \tag{4.5}$$

Proof. Without loss of generality, suppose that $0 = A0$.

Multiplying the equation of (4.3) by r_ε and integrating from 0 to T (for a fixed $n \in I\!N$), we deduce, via (4.4)

$$\int_0^T (Ar_\varepsilon, r_\varepsilon) dt + \frac{1}{2}\int_0^T (\|r_\varepsilon\|^2)' dt \le \varepsilon \int_0^T (r_\varepsilon'', r_\varepsilon) dt + \varepsilon^{n+1} \cdot M_n \left(\int_0^T \|r_\varepsilon\|^2 dt\right)^{1/2}$$

Denote by $|\cdot|$ the norm in $L^2(0,T;H)$. Since A is positive definitive, i.e. (1.1), then, the above inegality and the boundary conditions $r_\varepsilon(0) = r_\varepsilon(T) = 0$ imply

$$\omega \mid r_\varepsilon \mid \leq \varepsilon \mid r_\varepsilon'' \mid + \varepsilon^{n+1} \cdot M_n. \tag{4.6}$$

Next, one multiplies the equation in (4.3) by Ar_ε and one integrates over $[0, T]$. But A is symmetric, so, integrating by parts, we have

$$\int_0^T (Ar_\varepsilon, r_\varepsilon')dt = 0 \tag{4.7}$$

and

$$\int_0^T (Ar_\varepsilon, r_\varepsilon'')dt \leq -\omega \int_0^T \|r_\varepsilon'\|^2 dt \leq 0. \tag{4.8}$$

Then,

$$\int_0^T \|Ar_\varepsilon\|^2 dt = \varepsilon \int_0^T (r_\varepsilon'', Ar_\varepsilon)dt - \varepsilon^n \int_0^T (A\Pi_n(\tau) - \varepsilon u_n''(t), Ar_\varepsilon(t))dt,$$

hence (4.8) and (1.1) give us

$$\mid Ar_\varepsilon \mid^2 + \omega\varepsilon \mid r_\varepsilon' \mid^2 \leq \varepsilon^{n+1} M_n \mid Ar_\varepsilon \mid. \tag{4.9}$$

From this inequality, we infer

$$\mid Ar_\varepsilon \mid \leq \varepsilon^{n+1} M_n \tag{4.10}$$

and then, coming back to (4.9), one arrives at

$$\mid r_\varepsilon' \mid^2 \leq \varepsilon^{2n+1} M_n^2 / \omega. \tag{4.11}$$

Let us multiply the equation from (4.3) by r_ε'' and let us integrate over $[0, T]$, to obtain, with the aid of (4.8) and (4.4),

$$\varepsilon \mid r_\varepsilon'' \mid \leq \mid r_\varepsilon' \mid + \varepsilon^{n+1} M_n.$$

According to (4.11), this leads to

$$\varepsilon^2 \mid r_\varepsilon'' \mid^2 < 2M_n^2 \varepsilon^{2n+1}(1/\omega + \varepsilon), \tag{4.12}$$

therefore (4.6) implies

$$\omega \mid r_\varepsilon \mid \leq K_1 \varepsilon^{(2n+1)/2} + \varepsilon^{n+1} M_n, \tag{4.13}$$

where K_1 is a positive constant, independent of ε.

But, $\|r_\varepsilon(t)\|^2 = 2\int_0^t (r_\varepsilon(s), r_\varepsilon'(s))ds \leq 2 \mid r_\varepsilon \mid \mid r_\varepsilon' \mid$. Then, in view of (4.11) and (4.13), it follows that, there is a positive K_2, independent of ε, such that

$$\|r_\varepsilon(t)\|^2 \leq K_2 \varepsilon^{2n+1}, \quad t \in [0, T], \tag{4.14}$$

which leads to (4.5), as claimed.

Remark 4.2. Observe that $\mid Ar_\varepsilon \mid = O(\varepsilon^{n+1})$, $\mid r_\varepsilon' \mid = O(\varepsilon^{n+1/2})$, $\mid r_\varepsilon \mid = O(\varepsilon^{n+1/2})$.

Remark 4.3. For $n = 0$, we have $S_{0\varepsilon}(t) = u(t) + \Pi_0(\tau)$, hence, taking into account (3.20), we get

$$\|u_\varepsilon(t) - u(t)\| \leq \|u_\varepsilon(t) - S_{0\varepsilon}(t)\| + \|\Pi_0(\tau)\| \leq K_3\sqrt{\varepsilon} + \|b - u(T)\| e^{(t-T)/\varepsilon} \tag{4.15}$$

with $K_3 > 0$ independent of ε. It is clear that, for every $t \in [0, T]$,

$$u_\varepsilon(t) \to u(t), \quad \text{as } \varepsilon \to 0, \tag{4.16}$$

but

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)-u(t)\| \le K_3\sqrt{\varepsilon}+\|b-u(T)\| \to \|b-u(T)\|, \text{ as } \varepsilon\to 0. \tag{4.17}$$

Evidently, if $b=u(T)$, this supremum has the limit zero and it is of order $\sqrt{\varepsilon}$; therefore, the problem (A_ε) is regularly perturbed in $[0,T]$. If $b\neq u(T)$, then u_ε approaches u just outside of a neighbourhood $(\delta,T]$ of T (for some δ, $0<\delta<T$), because

$$\sup_{t\in[0,\delta]} \|u_\varepsilon(t)-u(t)\| \le K_3\sqrt{\varepsilon}+\|b-u(T)\|\cdot e^{(\delta-T)/\varepsilon} \to 0, \text{ as } \varepsilon\to 0. \tag{4.18}$$

In this case, (A_ε) is singularly perturbed.

5. APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

We are given two real Hilbert spaces V and H, such that $V\subseteq H$ and the inclusion mapping of V into H is continuous and densely defined. The space H is identified with its own dual H' and V' denotes the dual space of V. Then, the following inclusion relation holds:

$$V\subset H\subset V'.$$

It is known that, if A is a maximal monotone subset in $V\times V'$, then A_H defined by

$$D(A_H)=\{v\in V;\ Av\in H\}$$

$$A_H v = Av,\ \ (\forall)v\in D(A_H),$$

is a maximal monotone subset of $H\times H$.

We take $V=H_0^1(\Omega)$, $H=L^2(\Omega)$, $V'=H^{-1}(\Omega)$, where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma=\partial\Omega$. Let A be the linear operator given by

$$Au=-\Delta u=-\sum_{i=1}^{N}\frac{\partial^2 u}{\partial x_i^2}, \tag{5.1}$$

$$D(A)=H^2(\Omega)\cap H_0^1(\Omega). \tag{5.2}$$

Let a,b be two functions belonging to $D(A)$ and $f\in W^{n+1,2}(0,T;L^2(\Omega))$. Then the problems (A_0) and (A_ε) become

$$\begin{cases} \dfrac{\partial u}{\partial t}-\Delta_x u=-f, & \text{a.e. on } [0,T]\times\Omega\\ u=0 & \text{a.e. on } [0,T]\times\Gamma\\ u(0,\cdot)=a(\cdot) \end{cases} \tag{5.3}$$

and

$$\begin{cases} \varepsilon\dfrac{\partial^2 u_\varepsilon}{\partial t^2}-\dfrac{\partial u_\varepsilon}{\partial t}=-\Delta_x u_\varepsilon+f, & \text{a.e. on } [0,T]\times\Omega\\ u_\varepsilon=0 & \text{a.e. on } [0,T]\times\Gamma\\ u_\varepsilon(0,\cdot)=a(\cdot),\ \ u_\varepsilon(T,\cdot)=b(\cdot), \end{cases} \tag{5.4}$$

respectively. For a fixed $n\in\mathbb{N}$, let $S_{n\varepsilon}(t,x)$ be the n-th partial sum of the series (3.8) and $r_\varepsilon=u_\varepsilon-S_{n\varepsilon}$. Then, for $a,b\in H_0^1(\Omega)\cap H^2(\Omega)$, $f\in W^{n+1,2}(0,T;L^2(\Omega))$ and sufficiently small ε,

$$| r_\varepsilon |_{L^2([0,T]\times\Omega)}=O(\varepsilon^{n+1/2}) \tag{5.5}$$

and

$$\left| \frac{\partial r_\varepsilon}{\partial t} \right|_{L^2([0,T]\times\Omega)} = O(\varepsilon^n). \tag{5.6}$$

Here, r_ε verifies the linear problem

$$\begin{cases} \varepsilon \dfrac{\partial^2 r_\varepsilon}{\partial t^2} - \dfrac{\partial r_\varepsilon}{\partial t} = -\Delta r_\varepsilon - \varepsilon^n \left(\Delta \mathrm{II}_0 + \varepsilon \dfrac{\partial^2 u}{\partial t^2} \right), \text{ a.e. on } [0,T]\times\Omega \\ r_\varepsilon = 0 \text{ a.e. on } [0,T]\times\partial\Omega \\ r_\varepsilon(0,x) = r_\varepsilon(T,x) = 0 \text{ a.e. on } \Omega. \end{cases} \tag{5.7}$$

Since $\int_\Omega r_\varepsilon^2 dx = 2\int_0^t \int_\Omega r_\varepsilon \frac{\partial r_\varepsilon}{\partial t} dt dx$ a.e. $t \in [0,T]$, it follows that

$$\sup_{t\in[0,T]} \int_\Omega r_\varepsilon^2(t,x) dx = O(\varepsilon^{2n+1/2}).$$

In particular, for $n = 0$,

$$\sup_{t\in[0,T]} \left[\int_\Omega [u_\varepsilon(t,x) - u(t,x)]^2 dx \right]^{1/2} \le C\sqrt[4]{\varepsilon} + \int_\Omega [b(x) - u(T,x)]^2 dx.$$

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