

## EXISTENCE AND NON EXISTENCE RESULTS OF SOLUTIONS OF AN EQUATION WITH JUMPING NONLINEARITIES, 1

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**Abstract.** Let  $L$  be a selfadjoint operator with compact resolvent on  $L^2(\Omega)$  where  $\Omega$  is a bounded set in  $\mathbb{R}^n$ . In this paper, we discuss the existence and nonexistence results of solution of partial differential equation  $Lu = \alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u) + t\varphi$  where  $u^\pm = \max(0, \pm u)$  and  $\varphi$  is an eigenfunction of  $L$ . The results depend on scalar  $t$  and couple  $(\alpha, \beta)$  position towards the Fucik spectrum near the eigenvalue corresponding to  $\varphi$ .

### 1. INTRODUCTION.

In this work, we study the equation

$$Lu = Au + ug(\cdot, u) + r(\cdot, u) + t\varphi \tag{1}$$

where  $Au = \alpha u^+ - \beta u^-$ . We suppose that  $L : D(L) \subset L^2(\Omega) \rightarrow L^2(\Omega)$  is a selfadjoint operator with compact resolvent. Then  $D(L)$  is dense in  $L^2(\Omega)$ ,  $L$  is closed and the injection of  $(D(L), \|\cdot\|_{D(L)})$  in  $(L^2(\Omega), \|\cdot\|)$  is compact. We denote by  $\|u\|^2 = \int_{\Omega} u^2$ ,  $\langle u, v \rangle = \int_{\Omega} uv$  and  $\|u\|_{D(L)}^2 = \|u\|^2 + \|Lu\|^2$ . Let  $(\varphi_k)_k$  be the countable set of eigenfunctions of  $L$ , this set is an orthogonal basis of  $L^2(\Omega)$ . We consider a simple eigenvalue  $\lambda_k$  and we denote by  $\varphi$  the corresponding eigenfunction such that  $\|\varphi\| = 1$  and by  $I = ]\lambda_{k-1}, \lambda_{k+1}[$  (if  $\lambda_k$  is the smallest eigenvalue, we take  $\lambda_{k-1} = -\infty$  and if  $\lambda_k$  is the greatest eigenvalue, we take  $\lambda_{k+1} = \infty$ ). We suppose that  $(\alpha, \beta) \in I \times I$ . The functions  $g, r : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are Caratheodory function and there exists  $h \in L^2(\Omega)$  and  $d > 0$  such that  $|r(x, s)| \leq h(x)$  and  $0 \leq g(x, s) \leq d$ . Finally, the scalar  $d$  must be such that  $\max(\alpha + d, \beta + d) < \lambda_{k+1}$ .

Problems like (1) arise in a variety of asymmetric mechanical devices. For example, in [4], Lazer and McKenna proposed these as simple models describing vertical oscillations of long-span suspension bridges.

The equation (1) was studied by T. Gallouët and O. Kavian [1], [2] when  $\lim_{s \rightarrow \pm\infty} g(x, s) = g_{\pm} \in \mathbb{R}$  and  $\max(\alpha + g_+, \beta + g_-) < \lambda_{k+1}$ . In this paper, we suppose that the last limit doesn't exist. Under this assumption, (1) was treated by H. Asakawa [5] when  $\Omega = ]0, \pi[$  and  $Lu = -u''$  with Dirichlet conditions. That case is simple because the fucik spectrum and solutions of the equation  $-u'' = \alpha u^+ - \beta u^-$  are well known.

It was shown in [1] that when  $L$  is a selfadjoint operator with compact resolvent and  $\lambda_k$  is a simple eigenvalue, the Fucik spectrum  $\Sigma$  (i.e. the set of  $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$  such that the equation  $Lu = \alpha u^+ - \beta u^-$  has nontrivial solution in  $D(L)$ ) consists in  $I \times I$  of two nonincreasing curves passing through the point  $(\lambda_k, \lambda_k)$  and a couple  $(\alpha, \beta) \in I \times I$  belongs to  $\Sigma$  if and only if  $C(\alpha, \beta) * C(\beta, \alpha) = 0$  where  $C : I \times I \rightarrow \mathbb{R}$  is defined by :

$$\begin{cases} Lu &= \alpha u^+ - \beta u^- + C(\alpha, \beta)\varphi \\ \int u\varphi &= 1 \end{cases}$$

In the second section, we give some preliminaries results. The third section is devoted to existence and nonexistence results of solutions of (1).

## 2. PRELIMINARIES RESULTS.

**Theorem 1** [1] : Let  $q(x, s) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a caratheodory function such that  $q(x, 0) = 0$ . Suppose in addition that

$$\exists(\alpha, \beta) \in I \times I, \forall(s, t) \in \mathbb{R} \times \mathbb{R}, \alpha \leq \frac{q(x, s) - q(x, t)}{s - t} \leq \beta$$

then for each  $s \in \mathbb{R}$  and  $f \in L^2(\Omega)$  such that  $\int f\varphi = 0$ , the equation  $Lv = P_{\mathbb{R}\varphi^\perp}(q(\cdot, v + s\varphi)) + f$  admits a unique solution  $v$  in  $D(L)$  with  $\int v\varphi = 0$ . ( $P_{\mathbb{R}\varphi^\perp}$  : orthogonal projection on  $(\mathbb{R}\varphi)^\perp$ ).

**Theorem 2** [1] : For each  $(\alpha, \beta) \in I \times I$ , there exists a unique  $(u, C(\alpha, \beta)) \in D(L) \times \mathbb{R}$  such that

$$\begin{cases} Lu &= \alpha u^+ - \beta u^- + C(\alpha, \beta)\varphi \\ \int u\varphi &= 1 \end{cases}$$

the function  $C(\cdot, \cdot) : I \times I \rightarrow \mathbb{R}$  has the following properties :

1-  $C(\alpha, \alpha) = \lambda_k - \alpha$ .

2- If  $\varphi^- \equiv 0$ , then  $C(\alpha, \beta) = \lambda_k - \alpha$ .

3- If  $\|\varphi^-\| \neq 0$  and  $\|\varphi^+\| \neq 0$ , then  $C$  is decreasing in each variable.

**Lemma 1** [3] : Let  $\rho_1 > 0$  and  $\rho_2 < 0$  be such that

$$\rho_1 = \min \left\{ 4 \frac{(\min(\alpha, \beta) - \lambda_{k-1})^2}{|\beta - \alpha|}, 4 \frac{\lambda_{k+1} - \max(\alpha, \beta)}{4 + |\beta - \alpha| / (\lambda_{k+1} - \max(\alpha, \beta))} \right\}$$

$$\rho_2 = - \min \left\{ 4 \frac{(\lambda_{k+1} - \max(\alpha, \beta))^2}{|\beta - \alpha|}, 4 \frac{\min(\alpha, \beta) - \lambda_{k-1}}{4 + |\beta - \alpha| / (\min(\alpha, \beta) - \lambda_{k-1})} \right\}$$

if  $(C(\alpha, \beta) = 0$  and  $\int u\varphi \geq 0)$  or  $(C(\beta, \alpha) = 0$  and  $\int u\varphi \leq 0)$  then  $\|Lu - Au\|^2 \geq \rho_1 \langle Lu - Au, u \rangle$  and  $\|Lu - Au\|^2 \geq \rho_2 \langle Lu - Au, u \rangle$ .

The author suppose in [3] that the range of  $L$  is closed, which is a restriction. Without this assumption, we have the same conclusion using theorem 1. If  $C(\alpha, \beta) = 0$  and  $C(\beta, \alpha) = 0$ , then  $\|Lu - Au\|^2 \geq \rho_1 \langle Lu - Au, u \rangle$  and  $\|Lu - Au\|^2 \geq \rho_2 \langle Lu - Au, u \rangle$  for each  $u \in D(L)$ .

**Theorem 3** : For each  $(a, b) \in L^\infty(\Omega) \times L^\infty(\Omega)$  such that  $\lambda_{k-1} + \varepsilon \leq a(x), b(x) \leq \lambda_{k+1} - \varepsilon$  a.e. in  $\Omega$  for some  $\varepsilon > 0$ , there exists a unique couple  $(u, C(a, b)) \in D(L) \times \mathbb{R}$  such that

$$\begin{cases} Lu &= au^+ - bu^- + C(a, b)\varphi \\ \int u\varphi &= 1 \end{cases} \quad (2)$$

in addition, if  $a(x) \leq a_1(x)$  and  $b(x) \leq b_1(x)$  a.e. in  $\Omega$ , then  $C(a, b) \geq C(a_1, b_1)$ .

**Proof** : Let  $u = \varphi + v$  where  $v$  is such that  $\int v\varphi = 0$ . The function  $u$  is a solution of (2) if and only if the function  $v$  is a solution of  $Lv = P_{\mathbb{R}\varphi^\perp}(a(v+\varphi)^+ - b(v+\varphi)^-)$ . By theorem 1, the last equation has a unique solution  $v$ . Then  $u$  exists and it is unique, so we have  $C(a, b) = \lambda_k - \int (a(v+\varphi)^+ - b(v+\varphi)^-)\varphi$ . The constant  $C(a, b)$  is also given by [1] :

$$\begin{aligned} C(a, b) &= 2 \max_{v_1 \in H_1} \min_{v_2 \in H_2} J(v_1 + \varphi + v_2) \\ &= 2 \min_{v_2 \in H_2} \max_{v_1 \in H_1} J(v_1 + \varphi + v_2) \end{aligned} \quad (3)$$

where  $H_1 = \text{vect}(\dots, \lambda_{k-1})$ ,  $H_2 = \text{vect}(\lambda_{k+1}, \dots)$  and  $J : D(L) \rightarrow \mathbb{R}$  is such that  $J(u) = \frac{1}{2} \langle Lu, u \rangle - \frac{1}{2} \int a(u^+)^2 - \frac{1}{2} \int b(u^-)^2$ . The formula (3) allows us to show that  $C(\cdot, \cdot)$  is nonincreasing in each variable ■

**Theorem 4** : Let  $\rho \in I$ , there exists  $t(\rho) > 0$  such that for  $s \in [0, 1]$  and  $t \in \{|\tau| > t(\rho)\}$ , the equation

$$\begin{cases} Lu &= s(\alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u)) + (1-s)\rho u + t\varphi \\ \int u\varphi &= 0 \end{cases}$$

has no solution.

**Proof** (by contradiction) :  $\forall n \geq 0, \exists t_n \in \mathbb{R}, \exists u_n \in D(L)$  such that  $|t_n| \geq n$  and

$$\begin{cases} Lu_n &= s_n(\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) + (1-s_n)\rho u_n + t_n \varphi \\ \int u_n \varphi &= 0 \end{cases} \quad (4)$$

then  $t_n \varphi = Lu_n - s_n(\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) - (1-s_n)\rho u_n$ . So  $\alpha_n = \|u_n\|_{D(L)}$  goes to  $\infty$  and  $(t_n/\alpha_n)$  is bounded. Since the sequences  $(v_n = u_n/\alpha_n)$ ,  $(s_n)$ ,  $(t_n/\alpha_n)$  and  $(g(\cdot, u_n))$  are bounded, we can extract from them subsequences denoted in the same manner such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(s_n)$  tends to  $s$  in  $\mathbb{R}$ ,  $(t_n/\alpha_n)$  tends to  $C$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g'$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). It is easy to see that the function  $v$  is a solution of the equation

$$\begin{cases} Lv &= (s(\alpha + g') + (1-s)\rho)v^+ - (s(\beta + g') + (1-s)\rho)v^- + C\varphi \\ \int v\varphi &= 0 \end{cases}$$

By using theorem 1, we conclude that  $v = 0$  and  $C = 0$ . Then  $(v_n)$  converges to  $v$  in  $D(L)$ , so  $\|v\|_{D(L)} = 1$ . In contradiction with  $v = 0$  ■

### 3. EXISTENCE AND NONEXISTENCE RESULTS.

Let  $\rho \in I$ . For  $s \in [0, 1]$ , we define the equation

$$Lu = s(\alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u)) + (1-s)\rho u + t\varphi \quad (5)$$

We find (1) when we take  $s = 1$  in (5). A function  $u \in D(L)$  is a solution of (5) if and

only if  $u = T(s, u)$ , where  $T : [0, 1] \times L^2(\Omega) \rightarrow L^2(\Omega)$  is a completely continuous operator defined by :

$$T(s, u) = (L - \gamma I)^{-1} [s(\alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u)) + (1 - s)\rho u - \gamma u + t\varphi]$$

the scalar  $\gamma$  is such that  $(L - \gamma I)^{-1}$  is compact.

**Theorem 5** : If  $C(\beta + d, \alpha + d) > 0$ , then there exists  $t^* < 0$  such that for  $t < t^*$ , (1) has at least one solution  $u$  with  $\int u\varphi < 0$ . If  $C(\alpha + d, \beta + d) > 0$ , then there exists  $t^* > 0$  such that for  $t > t^*$ , (1) has at least one solution  $u$  with  $\int u\varphi > 0$ .

**Proposition 1** : Let  $\rho \in ]\lambda_{k-1}, \min(\alpha, \beta)[$ . If  $C(\beta + d, \alpha + d) > 0$ , then there exists  $R > 0$  such that for each  $s \in [0, 1]$  and  $u$  a solution of

$$\begin{cases} Lu &= s(\alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u)) + (1 - s)\rho u + t\varphi \\ \int u\varphi &< 0 \end{cases}$$

we have  $\|u\|_{D(L)} < R$ .

**Proof of proposition 1** (by contradiction) :  $\forall n > 0, \exists s_n \in [0, 1], \exists u_n \in D(L)$  such that  $\alpha_n = \|u_n\|_{D(L)} \geq n$  and

$$\begin{cases} Lu_n &= s_n(\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) + (1 - s_n)\rho u_n + t\varphi \\ \int u_n\varphi &< 0 \end{cases}$$

We can extract from  $(v_n = u_n/\alpha_n), (s_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(s_n)$  tends to  $s$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). The function  $v$  is a solution of the equation

$$\begin{cases} Lv &= (s(\alpha + g^*) + (1 - s)\rho)v^+ - (s(\beta + g^*) + (1 - s)\rho)v^- \\ \int v\varphi &\leq 0 \end{cases}$$

We show by theorem 1 that  $\int v\varphi \neq 0$ . If  $\int v\varphi < 0$ , then  $C(s(\beta + g^*) + (1 - s)\rho, s(\alpha + g^*) + (1 - s)\rho) = 0$ , that is impossible because  $C(s(\beta + g^*) + (1 - s)\rho, s(\alpha + g^*) + (1 - s)\rho) \geq C(\beta + d, \alpha + d) > 0$  ■

**Proof of theorem 5** : Let  $K = \{u \in L^2(\Omega) / \int u\varphi < 0 \text{ and } \|u\|_{D(L)} < R\}$  ( $R$  is given by proposition 1). The set  $K$  is open and bounded. Let  $t < -t(\rho)$  ( $t(\rho)$  is given by theorem 4). Hence for  $u \in \partial K$  and  $s \in [0, 1]$ , we have  $u \neq T(s, u)$ . The invariance by homotopy of topological degree of Leray-Schauder [7] allows us to conclude that  $D_{LS}(I - T(1, \cdot), K, 0) = D_{LS}(I - T(0, \cdot), K, 0)$ . For  $s = 0$ , the solution of the equation  $Lu = \rho u + t\varphi$  is such that  $\int u\varphi = t/(\lambda_k - \rho) < 0$  because  $t < 0$  and  $\lambda_k > \rho$ . Then  $u \in K$  and  $D_{LS}(I - T(0, \cdot), K, 0) = \pm 1$ . So (1) has at least one solution. The proof of second part of theorem 5 is carried out through the same steps as for the first part ■

**Theorem 6** : If  $C(\beta + d, \alpha + d) > 0$  and  $C(\alpha + d, \beta + d) > 0$ , then for each scalar  $t$ , (1) has at least one solution.

**Proposition 2** : Let  $\rho \in ]\lambda_{k-1}, \min(\alpha, \beta)[$ . Under the assumptions of theorem 6, there exists  $R > 0$  such that for  $s \in [0, 1]$  and  $u$  solution of (5), we have  $\|u\|_{D(L)} < R$ .

**Proof of proposition 2** (by contradiction) :  $\forall n > 0, \exists s_n \in [0, 1], \exists u_n \in D(L)$  such that  $\alpha_n = \|u_n\|_{D(L)} \geq n$  and

$$Lu_n = s_n(\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) + (1 - s_n)\rho u_n + t_n \varphi$$

We can extract from  $(v_n = u_n/\alpha_n)$ ,  $(s_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(s_n)$  tends to  $s$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). It is easy to verify that the function  $v$  is a solution of the equation

$$Lv = (s(\alpha + g^*) + (1 - s)\rho)v^+ - (s(\beta + g^*) + (1 - s)\rho)v^-$$

and that  $\int v\varphi \neq 0$ . If  $\int v\varphi < 0$ , then  $C(s(\beta + g^*) + (1 - s)\rho, s(\alpha + g^*) + (1 - s)\rho) = 0$ , in contradiction with  $C(s(\beta + g^*) + (1 - s)\rho, s(\alpha + g^*) + (1 - s)\rho) \geq C(\beta + d, \alpha + d) > 0$ . If  $\int v\varphi > 0$ , then  $C(s(\alpha + g^*) + (1 - s)\rho, s(\beta + g^*) + (1 - s)\rho) = 0$ , that is impossible because  $C(s(\alpha + g^*) + (1 - s)\rho, s(\beta + g^*) + (1 - s)\rho) \geq C(\alpha + d, \beta + d) > 0$  ■

We prove theorem 6 by using the invariance of topological degree of Leray-Schauder.

**Theorem 7** : If  $C(\alpha, \beta) = 0, C(\beta + d, \alpha + d) > 0$  and  $d < \rho_1$  ( $\rho_1$  is given by lemma 1), then there exists  $t^* > 0$  such that for  $t > t^*$ , (1) has no solution. If  $C(\beta, \alpha) = 0, C(\alpha + d, \beta + d) > 0$  and  $d < \rho_1$ , then there exists  $t^* < 0$  such that for  $t < t^*$ , (1) has no solution.

**Proof** (by contradiction) : Let  $C(\alpha, \beta) = 0, C(\beta + d, \alpha + d) > 0$  and  $d < \rho_1$ . Suppose that  $\forall n > 0, \exists t_n \geq n, \exists u_n \in D(L)$  such that

$$Lu_n = \alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n) + t_n \varphi$$

which is equivalent to  $Lu_n - (\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) = t_n \varphi$ . So  $\alpha_n = \|u_n\|_{D(L)}$  tends to  $\infty$  and  $(t_n/\alpha_n)$  is bounded. We can then extract from  $(v_n = u_n/\alpha_n)$ ,  $(t_n/\alpha_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(t_n/\alpha_n)$  tends to  $C \geq 0$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). The function  $v$  is a solution of the equation

$$Lv = \alpha v^+ - \beta v^- + g^* v + C\varphi$$

We prove by theorem 1 that  $\int v\varphi \neq 0$ . If  $\int v\varphi < 0$ , then  $C(\beta + g^*, \alpha + g^*) = C/\int v\varphi \leq 0$ , that is impossible because  $C(\beta + g^*, \alpha + g^*) \geq C(\beta + d, \alpha + d) > 0$ . If  $\int v\varphi > 0$ , then  $C(\alpha + g^*, \beta + g^*) = C/\int v\varphi \geq 0$ , but  $C(\alpha + g^*, \beta + g^*) \leq C(\alpha, \beta) = 0$ , so  $C = 0$  and  $v$  verifies  $Lv = \alpha v^+ - \beta v^- + g^* v$ . We obtain by lemma 1 that  $\int g^*(g^* - \rho_1)v \geq 0$ , so  $g^* v = 0$  because  $0 \leq g^* \leq d < \rho_1$ . Again by lemma 1, we have :

$$\|Lu_n - (\alpha u_n^+ - \beta u_n^-)\|^2 \geq \rho_1 \langle Lu_n - (\alpha u_n^+ - \beta u_n^-), u_n \rangle \quad (6)$$

(6) is equivalent to

$$\|u_n g(\cdot, u_n) + r(\cdot, u_n) + t_n \varphi\|^2 \geq \rho_1 \langle u_n g(\cdot, u_n) + r(\cdot, u_n) + t_n \varphi, u_n \rangle$$

thus

$$\begin{aligned} & \|u_n g(\cdot, u_n)\|^2 + \|r(\cdot, u_n)\|^2 + t_n^2 + 2 \langle u_n g(\cdot, u_n), r(\cdot, u_n) \rangle \\ & + 2t_n \langle u_n g(\cdot, u_n), \varphi \rangle + 2t_n \langle r(\cdot, u_n), \varphi \rangle \geq \\ & \rho_1 \langle u_n g(\cdot, u_n), u_n \rangle + \rho_1 \langle r(\cdot, u_n), u_n \rangle + \rho_1 t_n \langle \varphi, u_n \rangle \end{aligned} \quad (7)$$

dividing (7) by  $\alpha_n$ , we obtain :

$$\begin{aligned} t_n \left[ \frac{t_n}{\alpha_n} + 2 \langle v_n g(\cdot, u_n), \varphi \rangle + \frac{2}{\alpha_n} \langle r(\cdot, u_n), \varphi \rangle - \rho_1 \langle \varphi, v_n \rangle \right] \geq \\ \langle v_n g(\cdot, u_n), (\rho_1 - g(\cdot, u_n)) u_n \rangle - 2 \langle v_n g(\cdot, u_n), r(\cdot, u_n) \rangle \\ + \rho_1 \langle r(\cdot, u_n), v_n \rangle - \frac{1}{\alpha_n} \|r(\cdot, u_n)\|^2 \end{aligned}$$

since  $0 \leq g(\cdot, s) \leq d < \rho_1$  and  $|r(\cdot, s)| < h(\cdot)$ , we have :

$$\begin{aligned} t_n \left[ \frac{t_n}{\alpha_n} + 2 \langle v_n g(\cdot, u_n), \varphi \rangle + \frac{2}{\alpha_n} \langle r(\cdot, u_n), \varphi \rangle - \rho_1 \langle \varphi, v_n \rangle \right] \geq \\ -(2d + \rho_1) \|v_n\| \|h\| - \frac{1}{\alpha_n} \|h\|^2 \end{aligned} \quad (8)$$

when  $n$  goes to  $\infty$ , the left side of (8) tends to  $-\infty$ , in contradiction with the limit of the right side which is equal to  $-(2d + \rho_1) \|v\| \|h\|$ . The proof of the second part of theorem 7 is similar to that of the first part ■

**Theorem 8** : If  $C(\alpha, \beta) = 0$ ,  $C(\beta, \alpha) = 0$ ,  $d < \rho_1$  ( $\rho_1$  is given by lemma 1) and

$$\int \left\{ \liminf_{s \rightarrow \infty} N(\cdot, s)v^+ - \limsup_{s \rightarrow -\infty} N(\cdot, s)v^- \right\} + \frac{\rho_1}{\rho_1 - d} t \int v \varphi > \frac{d}{\rho_1 - d} \|v\| \|h\|$$

( $N(x, s) = sg(x, s) + r(x, s)$ ) for each nontrivial solution of  $Lv = \alpha v^+ - \beta v^-$ , then (1) has at least one solution.

**Proposition 3** : Under the assumptions of theorem 8, there exists  $R > 0$  such that for  $s \in [0, 1]$  and  $u$  a solution of the equation

$$Lu = \alpha u^+ - \beta u^- + s(ug(\cdot, u) + r(\cdot, u) + t\varphi) + (1 - s)\frac{d}{2}u \quad (9)$$

we have  $\|u\|_{D(L)} < R$ .

**Proof of proposition 3** (by contradiction) :  $\forall n > 0$ ,  $\exists s_n \in [0, 1]$ ,  $\exists u_n \in D(L)$  such that  $\alpha_n = \|u_n\|_{D(L)} \geq n$  and

$$Lu_n - (\alpha u_n^+ - \beta u_n^-) - \frac{d}{2}u_n = s_n(ug(\cdot, u_n) + r(\cdot, u_n) + t\varphi) - \frac{d}{2}u_n \quad (10)$$

By lemma 1, we have :

$$\left\| Lu_n - (\alpha u_n^+ - \beta u_n^-) - \frac{d}{2}u_n \right\|^2 \geq \frac{d^2}{4} \|u_n\|^2 + \left(1 - \frac{d}{\rho_1}\right) \|Lu_n - (\alpha u_n^+ - \beta u_n^-)\|^2$$

since  $|g(x, s) - \frac{d}{2}| \leq \frac{d}{2}$ , we have also :

$$\begin{aligned} \left\| u_n g(\cdot, u_n) + r(\cdot, u_n) + t\varphi - \frac{d}{2}u_n \right\|^2 \leq \frac{d^2}{4} \|u_n\|^2 + 2 \left\langle \left(g(\cdot, s) - \frac{d}{2}\right)u_n, r(\cdot, u_n) + t\varphi \right\rangle \\ + \|r(\cdot, u_n) + t\varphi\|^2 \end{aligned}$$

consequently

$$\begin{aligned} \left(1 - \frac{d}{\rho_1}\right) \|Lu_n - (\alpha u_n^+ - \beta u_n^-)\|^2 \leq 2s_n^2 \left\langle \left(g(\cdot, u_n) - \frac{d}{2}\right)u_n, r(\cdot, u_n) + t\varphi \right\rangle \\ + s_n^2 \|r(\cdot, u_n) + t\varphi\|^2 \end{aligned}$$

We conclude that  $(Lv_n - (\alpha v_n^+ - \beta v_n^-))$  tends to 0 in  $L^2(\Omega)$  where  $v_n = u_n/\alpha_n$ . Moreover it is possible to extract from  $(v_n)$ ,  $(s_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$  and there exists  $k \in L^2(\Omega)$  such that  $|v_n(x)| \leq k(x)$  a.e. in  $\Omega$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(s_n)$  tends to  $s$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). The function  $v$  is then a solution of the equation

$$Lv = \alpha v^+ - \beta v^- + sg^*v + (1 - s)\frac{d}{2}v$$

but  $Lv = \alpha v^+ - \beta v^-$  and  $v \neq 0$ , then  $s = 1$  and  $g^*v = 0$ . Multiplying (10) by  $v_n$ , we obtain :

$$\begin{aligned} & s_n \int N(\cdot, u_n)v_n + s_n t \int \varphi v_n + \frac{d}{2}(1 - s_n) \int u_n v_n \\ &= \langle Lu_n - (\alpha u_n^+ - \beta u_n^-), v_n \rangle \\ &\leq \frac{1}{\rho_1 \alpha_n} \|Lu_n - (\alpha u_n^+ - \beta u_n^-)\|^2 \\ &\leq \frac{2}{\rho_1 - d} s_n^2 \langle (g(\cdot, u_n) - \frac{d}{2})v_n, r(\cdot, u_n) + t\varphi \rangle + \frac{s_n^2}{\alpha_n} \|r(\cdot, u_n) + t\varphi\|^2 \end{aligned}$$

so

$$\begin{aligned} & s_n \int N(\cdot, u_n)v_n + s_n t \int \varphi v_n - \frac{2}{\rho_1 - d} s_n^2 \langle (g(\cdot, u_n) - \frac{d}{2})v_n, t\varphi \rangle \\ &\leq \frac{d}{\rho_1 - d} s_n^2 \|v_n\| \|h\| + \frac{s_n^2}{\alpha_n} \|r(\cdot, u_n) + t\varphi\|^2 \end{aligned}$$

When  $n$  goes to  $\infty$ , we obtain :

$$\liminf \int N(\cdot, u_n)v_n + \frac{\rho_1}{\rho_1 - d} t \int v\varphi \leq \frac{d}{\rho_1 - d} \|v\| \|h\|$$

since  $N(\cdot, u_n)v_n \geq r(\cdot, u_n)v_n \geq -h(\cdot)k(\cdot) \in L^1(\Omega)$ , we can use the fatou's lemma and we have :

$$\int \liminf_{s \rightarrow \infty} N(\cdot, s)v^+ - \limsup_{s \rightarrow -\infty} N(\cdot, s)v^- + \frac{\rho_1}{\rho_1 - d} t \int v\varphi \leq \frac{d}{\rho_1 - d} \|v\| \|h\|$$

which is in contradiction with assumptions ■

**Proof of theorem 8 :** When we take  $s = 1$  in (9), we find (1). A function  $u$  is a solution of (9) if and only if  $u = E(s, u)$  where

$$E(s, u) = (L - \gamma I)^{-1} \left[ \alpha u^+ - \beta u^- + s(ug(\cdot, u) + r(\cdot, u) + t\varphi) + (1 - s)\frac{d}{2}u - \gamma u \right]$$

The operator  $E$  is completely continuous and  $u \neq T(s, u)$  for  $u \in \partial B(R)$  and  $s \in [0, 1]$  ( $R$  is given by proposition 3). Hence  $D_{LS}(I - E(1, \cdot), B(R), 0) = D_{LS}(I - E(0, \cdot), B(R), 0)$ . Consider now the operator  $F : [0, 1] \times L^2(\Omega) \rightarrow L^2(\Omega)$  defined by :

$$F(s, u) = (L - \gamma I)^{-1} \left[ s(\alpha + \frac{d}{2})u^+ - s(\beta + \frac{d}{2})u^- + (1 - s)\delta u - \gamma u \right]$$

the scalar  $\delta$  is such that  $\lambda_{k+1} > \delta > \max(\alpha, \beta)$ . The operator  $F$  is completely continuous and  $F(1, u) = E(0, u)$ . If the equation  $u = F(s, u)$  has nontrivial solution  $u$ , then

$$Lu = s(\alpha + \frac{d}{2})u^+ - s(\beta + \frac{d}{2})u^- + (1 - s)\delta u$$

so  $\int u\varphi \neq 0$ . If  $\int u\varphi > 0$ , then  $C(s(\alpha + \frac{d}{2}) + (1 - s)\delta, s(\beta + \frac{d}{2}) + (1 - s)\delta) = 0$ , this

equality can't occur because  $C(s(\alpha + \frac{d}{2}) + (1-s)\delta, s(\beta + \frac{d}{2}) + (1-s)\delta) < C(\alpha, \beta) = 0$ . If  $\int u\varphi < 0$ , then  $C(s(\beta + \frac{d}{2}) + (1-s)\delta, s(\alpha + \frac{d}{2}) + (1-s)\delta) = 0$ , that is impossible because  $C(s(\beta + \frac{d}{2}) + (1-s)\delta, s(\alpha + \frac{d}{2}) + (1-s)\delta) < C(\beta, \alpha) = 0$ . Then the unique element  $u$  such that  $u = F(s, u)$  is 0. So  $D_{LS}(I - E(0, \cdot), B(R), 0) = D_{LS}(I - F(1, \cdot), B(R), 0) = D_{LS}(I - F(0, \cdot), B(R), 0) = \pm 1$ . Hence (1) has at least one solution ■

**Theorem 9** : If  $C(\alpha, \beta) = 0$ ,  $C(\beta, \alpha) = 0$  and  $g(x, s) = 0$  then for each scalar  $t$  such that  $|t| > \frac{b}{b-a} \|\sup_{s \in \mathbb{R}} |r(\cdot, s)|\|$  where  $b = \min(\lambda_{k+1} - \mu, \mu - \lambda_{k-1})$ ,  $a = |\alpha - \beta|/2$  and  $\mu = (\alpha + \beta)/2$ , (1) has no solution.

**Proof** : We use the Liapunov-Schmidt method. Since  $C(\alpha, \beta) = 0$  and  $C(\beta, \alpha) = 0$ , there exists a unique couple  $(w_1, w_2) \in D(L) \times D(L)$  such that

$$\begin{cases} Lw_1 = \alpha w_1^+ - \beta w_1^- \\ \int w_1 \varphi = 1 \end{cases} \quad \begin{cases} Lw_2 = \alpha w_2^+ - \beta w_2^- \\ \int w_2 \varphi = -1 \end{cases}$$

so the set  $G$  of solutions of the equation  $Lu = \alpha u^+ - \beta u^-$  is equal to  $\mathbb{R}^+ w_1 \cup \mathbb{R}^+ w_2$  and  $P_{\mathbb{R}\varphi} G = \mathbb{R}\varphi$ . Let  $t$  be a scalar such that (1) has a solution denoted  $u$  and let  $v \in G$  be such that  $\int v\varphi = \int u\varphi$ . The function  $(u - v)$  is then a solution of the equation

$$(L - \mu I)(u - v) = (A - \mu I)u - (A - \mu I)v + r(\cdot, u) + t\varphi \quad (11)$$

(11) is equivalent to

$$\begin{cases} (L - \mu I)(u - v) = P_{\mathbb{R}\varphi^\perp} [(A - \mu I)u - (A - \mu I)v + r(\cdot, u)] \\ 0 = \int [(A - \mu I)u - (A - \mu I)v + r(\cdot, u)] \varphi + t \end{cases} \quad (12)$$

Since the restriction  $(L - \mu I)|_{\mathbb{R}\varphi^\perp}$  is invertible with

$$\|(L - \mu I)|_{\mathbb{R}\varphi^\perp}^{-1}\|^{-1} = \text{dist}(\mu, \sigma(L) \setminus \{\lambda_k\}) = \min(\lambda_{k+1} - \mu, \mu - \lambda_{k-1}) = b$$

where  $\sigma(L)$  is the spectrum of  $L$  and since  $(A - \mu I)$  is lipschitzian with constant lipschitz  $|\alpha - \beta|/2 = a < b$ , we deduce from the first line of (12) that  $\|u - v\| \leq \frac{1}{b-a} \|\sup_{s \in \mathbb{R}} |r(\cdot, s)|\|$  and from the second line of (12) that  $|t| \leq \frac{b}{b-a} \|\sup_{s \in \mathbb{R}} |r(\cdot, s)|\|$  ■

**Theorem 10** : If  $C(\alpha, \beta) < 0$ , then there exists  $t^* < 0$  such that for  $t < t^*$ , (1) has at least one solution  $u$  with  $\int u\varphi > 0$ . If  $C(\beta, \alpha) < 0$ , then there exists  $t^* > 0$  such for  $t > t^*$ , (1) has at least one solution  $u$  with  $\int u\varphi < 0$

**Proposition 4** : Let  $p \in ]\max(\alpha, \beta), \lambda_{k+1}[$  and  $C(\alpha, \beta) < 0$ . There exists  $R > 0$  such that for  $s \in [0, 1]$  and  $u$  a solution of the equation

$$\begin{cases} Lu = s(\alpha u^+ - \beta u^- + ug(\cdot, u) + r(\cdot, u)) + (1-s)pu + t\varphi \\ \int u\varphi > 0 \end{cases}$$

we have  $\|u\|_{D(L)} < R$ .

**Proof of proposition 4** (by contradiction) :  $\forall n > 0, \exists s_n \in [0, 1], \exists u_n \in D(L)$  such that  $\alpha_n = \|u_n\|_{D(L)} \geq n$  and

$$\begin{cases} Lu_n = s_n(\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) + (1-s_n)pu_n + t\varphi \\ \int u_n \varphi > 0 \end{cases}$$

We can extract from  $(v_n = u_n/\alpha_n)$ ,  $(s_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(s_n)$  tends to  $s$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). The function  $v$  is then a solution of the equation

$$\begin{cases} Lv = (s(\alpha + g^*) + (1 - s)\rho)v^+ - (s(\beta + g^*) + (1 - s)\rho)v^- \\ \int v\varphi \geq 0 \end{cases}$$

so  $\int v\varphi \neq 0$ . If  $\int v\varphi > 0$ , then  $C(s(\alpha + g^*) + (1 - s)\rho, s(\beta + g^*) + (1 - s)\rho) = 0$ , in contradiction with  $C(s(\alpha + g^*) + (1 - s)\rho, s(\beta + g^*) + (1 - s)\rho) \leq C(\alpha, \beta) < 0$  ■

*Proof of theorem 10 :* Let  $K = \{u \in L^2(\Omega) / \int u\varphi > 0 \text{ and } \|u\|_{D(L)} < R\}$  ( $R$  is given by proposition 4). The set  $K$  is open and bounded. Let  $t < -t(\rho)$  ( $t(\rho)$  is given by theorem 4). Hence for  $u \in \partial K$  and  $s \in [0, 1]$ , we have  $u \neq T(s, u)$ . We conclude by the invariance of topological degree of Leray-Schauder that  $D_{LS}(I - T(1, \cdot), K, 0) = D_{LS}(I - T(0, \cdot), K, 0)$ . For  $s = 0$ , the solution of the equation  $Lu = \rho u + t\varphi$  is such that  $\int u\varphi = t/(\lambda_k - \rho) > 0$  because  $t < 0$  and  $\lambda_k < \rho$ . Then  $u \in K$  and  $D_{LS}(I - T(0, \cdot), K, 0) = \pm 1$ . So (1) has at least one solution. The proof of second part of theorem 10 is carried out through the same steps as for the first part ■

**Remark :** If  $C(\alpha, \beta) < 0$  and  $C(\beta + d, \alpha + d) > 0$  then, by using theorem 5 and theorem 10, we conclude that there exists  $t^* < 0$  such that for  $t < t^*$ , (1) has at least two solutions  $u_1$  and  $u_2$  with  $\int u_1\varphi > 0$  and  $\int u_2\varphi < 0$ . If  $C(\alpha + d, \beta + d) > 0$  and  $C(\beta, \alpha) < 0$ , then there exists  $t^* > 0$  such that for  $t > t^*$ , (1) has at least two solutions  $u_1$  and  $u_2$  with  $\int u_1\varphi > 0$  and  $\int u_2\varphi < 0$ .

**Theorem 11 :** *If  $C(\alpha, \beta) < 0$  and  $C(\beta + d, \alpha + d) > 0$ , then there exists  $t^* > 0$  such that for  $t > t^*$ , (1) has no solution. If  $C(\alpha + d, \beta + d) > 0$  and  $C(\beta, \alpha) < 0$ , then there exists  $t^* < 0$  such that for  $t < t^*$ , (1) has no solution.*

**Proof** (by contradiction) : Let  $C(\alpha, \beta) < 0$  and  $C(\beta + d, \alpha + d) > 0$ . Suppose that  $\forall n > 0, \exists t_n \geq n, \exists u_n \in D(L)$  such that

$$Lu_n = \alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n) + t_n \varphi$$

which is equivalent to  $Lu_n - (\alpha u_n^+ - \beta u_n^- + u_n g(\cdot, u_n) + r(\cdot, u_n)) = t_n \varphi$ . So  $\alpha_n = \|u_n\|_{D(L)}$  tends to  $\infty$  and  $(t_n/\alpha_n)$  is bounded. We can then extract from  $(v_n = u_n/\alpha_n)$ ,  $(t_n/\alpha_n)$  and  $(g(\cdot, u_n))$  subsequences such that  $(v_n)$  tends to  $v$  in  $L^2(\Omega)$ ,  $(Lv_n)$  tends to  $Lv$  weakly in  $L^2(\Omega)$ ,  $(t_n/\alpha_n)$  tends to  $C \geq 0$  in  $\mathbb{R}$  and  $(g(\cdot, u_n))$  tends to  $g^*$  weakly star in  $L^\infty(\Omega)$  (and weakly in  $L^2(\Omega)$ ). The function  $v$  is then a solution of the equation

$$Lv = \alpha v^+ - \beta v^- + g^* v + C\varphi$$

We prove by theorem 1 that  $\int v\varphi \neq 0$ . If  $\int v\varphi < 0$ , then  $C(\beta + g^*, \alpha + g^*) = C / \int v\varphi \leq 0$ , in contradiction with  $C(\beta + g^*, \alpha + g^*) \geq C(\beta + d, \alpha + d) > 0$ . If  $\int v\varphi > 0$ , then  $C(\alpha + g^*, \beta + g^*) = C / \int v\varphi \geq 0$ , that is impossible because  $C(\alpha + g^*, \beta + g^*) \leq C(\alpha, \beta) < 0$ . The proof of second part of theorem 11 is similar to that of the first part ■

An open problem is to study (1) when  $\lambda$  is not a simple eigenvalue of  $L$  and  $\varphi$  is an eigenvector associated to  $\lambda$ . This problem is difficult because we don't have much information about the Fucik Spectrum.

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