LIBERTAS MATHEMATICA, VOL. XIX (1999)

ADJOINT VARIABLES FOR HIGHER-ORDER EQUATIONS. APPLICATIONS TO SPINNING PARTICLE

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Abstract. *We obtain a type of first integrals for ODE by using a generalization of the method of adjoint variables to higher-order systems. The case of classical spinning particle is completely discussed in order to solve the associated inverse problem.*

Subject Classification: 34A34, 34A30, 70D99.
Key words and phrases: conservation law, adjoint system, spinning particle.

1. INTRODUCTION

Until now, the method of *adjoint*(or *added*) *variables* was used only for systems of differential equations which involve the derivatives of first and second order. So, for second order ODE see [4], [5] and for second order PDE see [1], [2]. For other informations about this approach see [9, chapter 3].

In the first paragraph we present the generalization of this method to higher-order ODE. Next is considered the linear case and more particularly the linear autonomous case. As example is given the equations of motion for spinning particle. By using the obtained first integrals the coresponding inverse problem is resolved.

Acknowledgement Special thanks are offered to the author's Ph.D. adviser , prof. Radu Miron, for introduction in higher order dynamics.

2. THE ADJOINT VARIABLES APPROACH

Let us consider a general system of k-order ODE, with k and n two natural numbers:

$$F^j\left(t, x, x_1, \ldots, x_k\right) = 0, 1 \leq j \leq n \tag{1.1}$$

with the unknow function $x = \left(x^i\right)_{1 \leq i \leq n}$ and:

$$x^i_\alpha = \frac{d^\alpha x^i}{dt^\alpha}, 1 \leq \alpha \leq k. \tag{1.2}$$

A *conservation law*(or *first integral* or *conserved quantity*) for the system (1.1) is a function $\mathcal{F} = \mathcal{F}\left(t, x, x_1, \ldots, x_{k-1}\right)$ satisfying:

$$\frac{d\mathcal{F}}{dt} = 0, \mathrm{mod}\,(1.1) \tag{1.3}$$

where mod (1.1) means "on the solutions of (1.1)" and $\frac{d}{dt}$ denotes the total differentiation with respect to t: $\frac{d}{dt} = \frac{\partial}{\partial t} + x_1^i \frac{\partial}{\partial x^i} + \ldots + x_k^i \frac{\partial}{\partial x_{k-1}^i}$.

For $\xi = \left(\xi^i\right)_{1 \leq i \leq n}$ let the point-transformation:

$$\tilde{x}^i = x^i + \varepsilon \xi^i \tag{1.4}$$

with ε being a real number with sufficiently small absolute value. The Frechet derivative of F^j is:

$$dF^j(x)(\xi) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}\left(\tilde{F}^j - F^j\right) \tag{1.5}$$

where $\tilde{F}^j = F^j(t, \tilde{x}, \ldots, \tilde{x}_k)$. Then:

$$dF^j(x)(\xi) = \sum_{\alpha=1}^{k} \frac{\partial F^j}{\partial x_{k-\alpha+1}^i} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}}. \tag{1.6}$$

We search for (1.1) a conservation law with expression:

$$\mathcal{F} = \sum_{\alpha=1}^{k} (-1)^{\alpha+1} T_{(\alpha)i} \frac{d^{\alpha-1}\xi^i}{dt^{\alpha-1}} - K \tag{1.7}$$

with $\frac{d^0 f}{dt^0} = f$ and $K = K(t, x, \ldots, x_{k-1})$. We require that:

$$\frac{d\mathcal{F}}{dt} = \mu_j \sum_{\alpha=1}^{k+1} \frac{\partial F^j}{\partial x_{k-\alpha+1}^i} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}} - \frac{dK}{dt} \tag{1.8}$$

with $\mu = \left(\mu_j\right)_{1 \leq j \leq n}$.

From (1.7) and (1.8) it results by comparing the coefficients of $\frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}}$:

$$T_{(\alpha)i} = \frac{d}{dt} T_{(\alpha-1)i} + (-1)^{\alpha+1} \mu_j \frac{\partial F^j}{\partial x_{k-\alpha+1}^i}, 1 \leq \alpha \leq k \tag{1.9}$$

and we obtain the main result of the paper:

Theorem 1 *If the functions ξ, μ of (t, x) satisfy the system:*

$$\frac{d}{dt} T_{(k)i} + (-1)^k \mu_j \frac{\partial F^j}{\partial x^i} = 0 \tag{1.10a}$$

$$\mu_j \sum_{\alpha=1}^{k+1} \frac{\partial F^j}{\partial x_{k-\alpha+1}^i} \frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}} = \frac{dK}{dt} \tag{1.10b}$$

on the solutions of (1.1) with $K = K(t, x, \ldots, x_{k-1})$ a given function then $\mathcal{F}$ given by (1.7) is conservation law for (1.1).

Remark 1 *(i) If F^j does not depend of x then (1.10a) becomes: $T_{(k)i} = const.$*
(ii) The relation (1.10b) is exactly:

$$\mu_j dF^j(x)(\xi) = 0 \tag{1.11}$$

and then the equations (1.9) + (1.10a) are:

$$(dF(x))_i^*(\mu) = 0 \tag{1.12}$$

where $(dF(x))_i^*(\mu)$ is the adjoint of the linear operator $dF(x)$:

$$(-1)^k (dF(x))_i^*(\mu) = \frac{d^k}{dt^k}\left(\frac{\partial F^j}{\partial x_k^i}\mu_j\right) - \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial F^j}{\partial x_{k-1}^i}\mu_j\right) + \ldots +$$
$$+(-1)^{k-1}\frac{d}{dt}\left(\frac{\partial F^j}{\partial x_1^i}\mu_j\right) + (-1)^k \frac{\partial F^j}{\partial x^i}\mu_j. \tag{1.13}$$

Recall that the vectorial operator $F = (F^j)$ is called *self-adjoint* if the Frechet derivative of F is self-adjoint and this property is the necessary and sufficient condition for the system $F^j = 0$ to be yield by a variational principle i.e. there exists a Lagrangian L such that the given system is exactly th Euler-Lagrange system of L.

3. THE LINEAR CASE

Let us suppose that F^j is linear:

$$F^j(x) = A_{(0)i}^j x_k^i + \ldots + A_{(k)i}^j x^i \tag{2.1}$$

with $A_{(\alpha)i}^j = A_{(\alpha)i}^j(t)$, $1 \leq i,j \leq n$, $0 \leq \alpha \leq k$. Therefore:

$$\frac{\partial F^j}{\partial x_{k-\alpha}^i} = A_{(\alpha)i}^j \tag{2.2}$$

and then: $dF^j(x)(\xi) = F^j(\xi)$. One obtain:

$$(dF(x))_i^*(\mu) = F_i^*(\mu) = \frac{d^k}{dt^k}\left(A_{(0)i}^j\mu_j\right) - \frac{d^{k-1}}{dt^{k-1}}\left(A_{(1)i}^j\mu_j\right) + \ldots +$$
$$+(-1)^{k-1}\frac{d}{dt}\left(A_{(k-1)i}^j\mu_j\right) + (-1)^k A_{(k)i}^j\mu_j \tag{2.3}$$

and then, for linear systems the theorem becomes:

Proposition 1 *If ξ and μ satisfy:*

$$F_i^*(\mu) = 0 \tag{2.4a}$$

$$\mu_j F^j(\xi) = \frac{dK}{dt} \tag{2.4b}$$

then $\mathcal{F}$ given by (1.7) is conservation law for (1.1) with $\mathcal{F}$ given by (2.1).

More particularly if the coefficients $A_{(\alpha)i}^j$ are constants, that is the case of linear autonomous systems, then:

Corollary 1 *If ξ and μ satisfy:*

$$A_{(0)i}^j\frac{d^k\mu_j}{dt^k} - A_{(1)i}^j\frac{d^{k-1}\mu_j}{dt^{k-1}} + \ldots + (-1)^{k-1}A_{(k-1)i}^j\frac{d\mu_j}{dt} + (-1)^k A_{(k)i}^j\mu_j = 0 \tag{2.5a}$$

$$\mu_j F^j(\xi) = \frac{dK}{dt} \tag{2.5b}$$

then $\mathcal{F}$ given by (1.7) is first integral for (2.1).

4. AN EXAMPLE: THE SPINNING PARTICLE

The motion of a particle in rotation around a center in translation is given by the equations([3]):

$$F^i = \frac{d^4x^i}{dt^4} + \frac{d^2x^i}{dt^2} = 0, 1 \leq i \leq 3 \tag{3.1}$$

which is a linear system. The associated system (2.5) is:

$$\frac{d^4\mu_i}{dt^4} + \frac{d^2\mu_i}{dt^2} = 0 \tag{3.2a}$$

$$\mu_j \left(\frac{d^4\xi^j}{dt^4} + \frac{d^2\xi^j}{dt^2} \right) = \frac{dK}{dt} \tag{3.2b}$$

and for $K = 0$ we have two solutions:

$$\mu_i = \cos t, \xi^i = x^i \tag{3.3a}$$

$$\mu_i = \sin t, \xi^i = x^i \tag{3.3b}$$

with coresponding conservation laws:

$$\mathcal{F}_1^i = \frac{d^3x^i}{dt^3} \cos t + \frac{d^2x^i}{dt^2} \sin t \tag{3.4a}$$

$$\mathcal{F}_2^i = \frac{d^3x^i}{dt^3} \sin t - \frac{d^2x^i}{dt^2} \cos t. \tag{3.4b}$$

The system (3.1) is self-adjoint and then we are interested in finding the associated Lagrangian. From (3.1) it results the conservation law:

$$C^i = \frac{d^3x^i}{dt^3} + \frac{dx^i}{dt} \tag{3.5}$$

which, in order to eliminate the variable t, yields the conservation law:

$$\Psi^i = \left(C^i\right)^2 - \left(\mathcal{F}_1^i\right)^2 - \left(\mathcal{F}_2^i\right)^2. \tag{3.6}$$

A straightforward computation gives:

$$\Psi^i = \left(\frac{dx^i}{dt}\right)^2 - \left(\frac{d^2x^i}{dt^2}\right)^2 + 2\frac{dx^i}{dt}\frac{d^3x^i}{dt^3} \tag{3.7}$$

and then we have the conservation law:

$$H = \frac{1}{2}\left(\Psi^1 + \Psi^2 + \Psi^3\right) =$$

$$= \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2 - \frac{1}{2}\sum_{i=1}^{3}\left(\frac{d^2x^i}{dt^2}\right)^2 + \sum_{i=1}^{3}\frac{dx^i}{dt}\frac{d^3x^i}{dt^3} \tag{3.8}$$

which is exactly the Hamiltonian for (3.1)([3]). The associate Lagrangian is([3]):

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2 - \frac{1}{2}\sum_{i=1}^{3}\left(\frac{d^2x^i}{dt^2}\right)^2 \tag{3.9}$$

(that is (3.1) are the Euler-Lagrange equations for L) a result very important from the point of view of Inverse Problem of Analytical Mechanics([8]). Thus we solve on this way the inverse problem for spinning particle.

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